CS 174: Combinatorics and Discrete Probability Spring 2016

Final Exam

Monday May 9

Notes: *Please read all the questions carefully and ask me for any clarifications. Some questions are harder than others; if you are stuck, I suggest you move on to a different question and later on come back to the one on which you were stuck. Please write clearly and legibly. Good Luck!*

Name (10 points!):

SID:

Problem 1. *(Total 15 points)*
Warm-up questions.

(a) What is a Las Vegas algorithm? Give one example that was covered in this class.

(b) What is a Monte Carlo algorithm? Give one example that was covered in this class.

(c) Describe how to turn a Monte Carlo algorithm into a Las Vegas algorithm (under "favorable" conditions) and give an example.

(d) Give an example of a Monte Carlo algorithm that is difficult to convert into a Las Vegas algorithm and explain where the difficulty arises.

(e) Explain how you can use Wald's equation in analyzing the expected running time of a Las Vegas algorithm constructed from a Monte Carlo algorithm as in part (c).

Problem 2. *(Total 15 points)*

Consider a sequence of $n > k$ flips of an unfair coin with $\Pr(H) = p$. You may ignore rounding issues for this problem.

(a) What is the probability of getting exactly k heads?

(b) What the expected number of times you get k heads in a row? (Count overlaps, so the sequence of all heads has $n - k + 1$ copies of "k heads in a row.")

(c) Show that if $k = [\ln(n) - 3\ln(\ln(n))]/\ln(1/p)$ then for sufficiently large n, the probability of having no streak of k heads in a row converges to 0.

(d) Construct a Markov Chain and use it to compute the probability that there are at least 4 consecutive heads ($k = 4$). (Just show the transition matrix and a linear-algebraic equation for the solution. You do **not** need to evaluate this numerically.)

Problem 3. *(Total 15 points)*

Consider throwing n balls randomly into $2n$ bins.

(a) What is the exact probability that the first n bins are empty and the remaining n bins have 1 ball each?

(b) Redo part (a) using the Poisson Approximation for balls and bins to upper bound this value.

(c) Show that your result in part (b) is indeed an upper bound for part (a). You may use the bounds that result from Stirling's approximation:

$$\sqrt{2\pi n}^{n+1/2}\sqrt{n}e^{-n} \leq n! \leq en^{n+1/2}e^{-n}.$$

Problem 4. *(Total 15 points)*

Consider a graph G which has exactly m k-cycles.

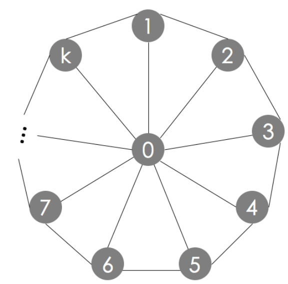
(a) Show that there exists a 2-coloring of the nodes of G so that at most $m2^{-k+1}$ of the k-cycles are monochromatic.

(b) Construct a Las Vegas algorithm for finding such a coloring. Show that it runs in expected polynomial time in m and n (the number of vertices).

(c) Derandomize the algorithm in the previous part to give a deterministic algorithm. (Just describe the algorithm. You do not need to analyze the running time.)

Problem 5. *(Total 15 points)*

Consider a random walk on the wheel graph with a central node 0 inside a k-cycle of nodes $1, \ldots, k$, where the edges consist of



- $(0, i)$ for all $1 \leq i \leq k$,
- $(i, i+1)$ for all $1 \leq i < k$,
- and $(k, 1)$.

(a) What is the stationary distribution for this random walk?

(b) Compute the expected return time for a random walk starting at 0.

(c) Compute $h_{1,0}$ using your result from the previous part.

(d) Prove that $h_{0,1} = \Omega(k)$. Hint: use symmetry. (You do not need to compute it explicitly.)

(e) Prove that the cover time for this graph is bounded by $kh_{0,1} + 3k$.

(f) Improve your bound for the cover time to $O(k \log k)$. (Hint: Combine coupon collecting with the standard argument from the previous part.)

Problem 6. *(Total 15 points)*

An ordinary deck of cards is randomly shuffled and then the cards are exposed one at a time. At some time before all the cards have been exposed you must say "next", and if the next card exposed is a spade then you win and if not then you lose. Is there a strategy that is better than simply saying next immediately?

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