CS 70 Discrete Mathematics for CS
Fall 2003 Wagner

Midterm 1

PRINT your name: ______________________________, ______________________________
(last) (first)

SIGN your name: ____

PRINT your username on `cory.eecs`: ____

WRITE your section number (101 or 102): ___________________________

This exam is open-book, open-notes. No calculators are permitted. Do all your work on the pages of this examination. If you need more space, you may use the reverse side of the page, but try to use the reverse of the same page where the problem is stated.

You have 80 minutes. There are 4 questions, worth 25 points each (100 points total). The questions are of varying difficulty, so avoid spending too long on any one question.

Do not turn this page until the instructor tells you to do so.

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Total	

Problem 1. [True or false] (25 points)

Circle TRUE or FALSE. You do not need to justify your answers on this problem.

N denotes the set of natural numbers, $\{0,1,2,\ldots\}$. **Z** denotes the integers, $\{\ldots,-2,-1,0,1,2,\ldots\}$.

(a) TRUE or FALSE: If the implication $P \implies Q$ is true, then its converse is guaranteed to be true, too.

(b) TRUE or FALSE: $\forall w \in \mathbf{Z}.\ \exists x \in \mathbf{Z}.\ \forall y \in \mathbf{Z}.\ \exists z \in \mathbf{Z}.\ w+x=y+z$.

(c) TRUE or FALSE: $\exists x \in \mathbf{N}.\ \forall p \in \mathbf{Z}.\ p>5 \implies x^2 \equiv 1 \pmod{p}$.

(d) TRUE or FALSE: $\forall p \in \mathbf{Z}.\ p>5 \implies \exists x \in \mathbf{N}.\ x^2 \equiv 1 \pmod{p}$.

(e) TRUE or FALSE: If m is any natural number satisfying $m \equiv 1 \pmod{2}$, then the equation $2048x \equiv 1 \pmod{m}$ is guaranteed to have a solution for x.

Problem 2. [Proof by Induction] (25 points)

Prove by induction that

$$\sum_{i=1}^{n} \frac{i(i-1)}{2} = \frac{(n+1)n(n-1)}{6}$$

holds for all $n \in \mathbf{N}$.

Problem 3. [Proofs] (25 points)

Definition: A 2-party cake-cutting protocol is called *equalizing* if it satisfies the following property: If a denotes the worth (by Alice's measure) of the piece Alice receives, and b denotes the worth (by Bob's measure) of the piece Bob receives, then $a = b$.

(a) TRUE or FALSE: Every envy-free 2-party cake-cutting protocol is equalizing.

(b) Prove your answer to part (a).

(c) Prove the following: If x is positive and irrational, then $\sqrt{x}$ is irrational, too.

Problem 4. [Strings] (25 points)

Let $\{0,1\}^*$ denote the set of all binary strings. Write $y \cdot z$ for the concatenation of the strings y and z.

Prove that every string $x \in \{0,1\}^*$ can be written in the form $x = y \cdot z$ where the number of 0's in y is the same as the number of 1's in z. Empty strings are allowed.

(For instance, 01001 can be split as $01 \cdot 001$; $111011 = 11101 \cdot 1$; and $00000 = \cdot 00000$.)

Hint: It is possible to prove this using strong induction over $\mathbf{N}$.