EECS 120 **Fall 2004**

Midterm 1 Solutions

• The exam is for one hour and 50 minutes.

• The maximum score is 100 points. The maximum score for each part of each problem is indicated.

• The exam is closed-book and closed-notes. Calculators, computing, and communication devices are NOT permitted.

• Two double-sided sheets of notes are allowed. These should be legible to normal eyesight, i.e. the lettering should not be excessively small.

• No form of collaboration between students is allowed.

1. *(10 points)* State whether the following are true or false. In each case, give a brief explanation. A correct answer without a correct explanation gets 2 points. A correct answer with a correct explanation gets 5 points.

   (a) Let $x(t)$ be a continuous time signal. Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ denote the respective ouputs of a causal linear time invariant system to the inputs $x(t)$, $x^2(t)$, and $x^3(t)$. Then $y_3(t)$ can be determined from $y_1(t)$ and $y_2(t)$.

   (b) If $x[n]$ is a nonnegative sequence with discrete time Fourier transform $X(e^{j\omega})$, then

$$\sum_{n=-\infty}^{\infty} x[n] = \frac{1}{2\pi} \int_{2\pi} \mid X(e^{j\omega}) \mid d\omega \ .$$

   Solution :

   (a) False. Consider for example $X(j\omega) = \begin{cases} 1 & \text{if } \mid \omega \mid \leq 1 \\ 0 & \text{otherwise} \end{cases}$. Then, to find out $y_3(t)$ we would need to know something about the transfer function of the system for frequencies in the range $2 <\mid \omega \mid< 3$. However the given information can at most tell us about the transfer function in the range of frequencies $\mid \omega \mid \leq 2$.

   (b) False. Take $x[n] = \delta[n]+\delta[n-1]$. Then $X(e^{j\omega}) = 1+e^{-j\omega}$. We have $\sum_{n=-\infty}^{\infty} x[n] = 2$. Note that $\mid 1 + e^{-j\omega} \mid< 2$ except when ω is an integer multiple of 2π. Thus

$$\frac{1}{2\pi} \int_{2\pi} \mid X(e^{j\omega}) \mid d\omega < 2 \ ,$$

   so the claimed equality cannot hold.

2. *(10 points)*

Let $x[n]$ and $y[n]$ be periodic with period 3, with

$$x[n] = \begin{cases} 1 & \text{if } n = -1 \\ 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \end{cases} ,$$

and

$$y[n] = \begin{cases} -1 & \text{if } n = -1 \\ 2 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \end{cases} .$$

Let $z[n]$ be the periodic sequence of period 3 that is the periodic convolution of $x[n]$ and $y[n]$, i.e.

$$z[n] = \sum_{l \in <3>} x[l]y[n-l] .$$

Determine $z[n]$.

Solution : We have

$$z[n] = \sum_{l \in <3>} x[l]y[n-l] .$$

Hence

$$\begin{aligned} z[-1] &= x[-1]y[0] + x[0]y[-1] + x[1]y[-2] \\ &= 1 \cdot 2 + 2 \cdot (-1) + 1 \cdot 1 \\ &= 1 , \\ z[0] &= x[-1]y[1] + x[0]y[0] + x[1]y[-1] \\ &= 1 \cdot 1 + 2 \cdot 2 + 1 \cdot (-1) \\ &= 4 , \\ z[1] &= x[-1]y[2] + x[0]y[1] + x[1]y[0] \\ &= 1 \cdot (-1) + 2 \cdot 1 + 1 \cdot 2 \\ &= 3 . \end{aligned}$$

Since $z[n]$ is periodic with period 3, this determines $z[n]$.

3. *(10 points)*

Let

$$\Lambda(t) = \begin{cases} 1- \mid t \mid & \text{if } \mid t \mid \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Let

$$x(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n) \cos(\omega_0 t) .$$

Find the Fourier transform of $x(t)$.

Hint : The function

$$z(t) = \begin{cases} 1 & \text{if } \mid t \mid \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

has Fourier transform

$$Z(j\omega) = \text{sinc}(\frac{\omega}{2\pi}) \ .$$

Solution : The Fourier transform of $\Lambda(t)$ is $\text{sinc}^2(\frac{\omega}{2\pi})$. Since

$$\sum_{n=-\infty}^{\infty} \Lambda(t-2n) = \Lambda(t) * (\sum_{n=-\infty}^{\infty} \delta(t-2n))$$

and the Fourier transform of $\sum_{n=-\infty}^{\infty} \delta(t-2n)$ is $\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\pi)$, the Fourier transform of $\sum_{n=-\infty}^{\infty} \Lambda(t-2n)$ is $\pi \sum_{k=-\infty}^{\infty} \text{sinc}^2(\frac{k}{2})\delta(\omega - k\pi)$. This can be simplified to

$$\pi\delta(\omega) + \sum_{k \ odd} \frac{4}{\pi k^2}\delta(\omega - k\pi) \ .$$

Since the Fourier transform of $\cos(\omega_0 t)$ is $\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$, we get

$$\begin{aligned} X(j\omega) &= \frac{\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \\ &\quad + \sum_{k \ odd} \frac{2}{\pi k^2}\left[\delta(\omega - k\pi - \omega_0) + \delta(\omega - k\pi + \omega_0)\right] \ . \end{aligned}$$

4. *(10 points)* Let $x[n]$ be a periodic sequence with period N. Assume $N = 3K$ for some integer K. Let a_k denote the discrete time Fourier series coefficients of $x[n]$. If $a_k = 0$ when k is not a multiple of 3, show that $x[n]$ must also be periodic with period K.

   Solution : By the definition of the discrete time Fourier series coefficients, we have

$$a_k = \frac{1}{N} \sum_{n \in <N>} x[n] e^{-jk\frac{2\pi}{N}n} \ .$$

   Also,

$$x[n] = \sum_{k \in <N>} a_k e^{jk\frac{2\pi}{N}n} \ .$$

   Now consider $x[n+K]$. We have

$$\begin{aligned} x[n+K] &= \sum_{k \in <N>} a_k e^{jk\frac{2\pi}{N}(n+K)} \\ &= \sum_{k \in <N>} a_k e^{jk\frac{2\pi}{N}n} e^{jk\frac{2\pi}{3}} \\ &\overset{(a)}{=} \sum_{k=3l \in <N>} a_k e^{jk\frac{2\pi}{N}n} e^{jk\frac{2\pi}{3}} \\ &= \sum_{k=3l \in <N>} a_k e^{jk\frac{2\pi}{N}n} \\ &\overset{(b)}{=} \sum_{k \in <N>} a_k e^{jk\frac{2\pi}{N}n} \\ &= x[n] \ . \end{aligned}$$

   Here step (a) and step(b) use the given property that $a_k = 0$ unless k is a multiple of 3. We have thus shown that $x[n]$ is also periodic with period K.

5. *(5 + 5 points)* Consider the causal linear time-invariant system whose input and output are related by the difference equation

$$y[n] + \frac{1}{3}y[n-1] = x[n] + x[n-2] - 3x[n-5] \ .$$

(a) Find the transfer function of the system.

(b) Find the output of this system for the input

$$x[n] = (-1)^n \text{ for all } n \ .$$

Solution :

(a) The corresponding discrete time Fourier transform equation is

$$(1 + \frac{1}{3}e^{-j\omega})Y(e^{j\omega}) = (1 + e^{-2j\omega} - 3e^{-5j\omega})X(e^{j\omega}) \ .$$

This yields the transfer function of the system as

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + e^{-2j\omega} - 3e^{-5j\omega}}{1 + \frac{1}{3}e^{-j\omega}} \ .$$

(b) The input given is the discrete time pure tone

$$x[n] = e^{j\pi n} \ .$$

Substituting $\omega = \pi$ in the transfer function, we see the corresponding output must be

$$y[n] = H(e^{j\pi})x[n] = \frac{1 + e^{-2j\pi} - 3e^{-5j\pi}}{1 + \frac{1}{3}e^{-j\pi}}e^{j\pi n} = \frac{5}{2/3}(-1)^n = \frac{15}{2}(-1)^n \ ,$$

for all n.

6. *(5 + 5 + 5 points)* Consider the continuous time system whose output $y(t)$ for the input $x(t)$ is given by

$$y(t) = x(t - (\int_t^{t+1} x(u)du)^2) \ .$$

Is it :

(a) linear ?

(b) causal ?

(c) BIBO stable ?

In each case explain your answers briefly. There should be no ambiguity about which part of the problem you are answering. A correct answer with an incorrect explanation will get at most 2 points.

Solution

(a) No, the system is not linear. The input $x_1(t) = 1$ has corresponding output $y_1(t) = 1$ and the input $x_2(t) = u(t)$ has corresponding output $y_2(t) = u(t-1)$. However, the input $x(t) = 1 + u(t)$ has corresponding output

$$y(t) = 1 + u(t - \frac{1}{4}) \neq y_1(t) + y_2(t) \ .$$

(b) No, the system is not causal. If $x_1(t) = x_2(t)$ for $t < t_0$, the corresponding outputs $y_1(t)$ and $y_2(t)$ respectively need not satisfy $y_1(t) = y_2(t)$ for $t < t_0$. For example, consider $x_1(t) =\mid t \mid u(-t)$ and $x_2(t) =\mid t \mid u(-t) + u(t)$. We have $x_1(t) = x_2(t)$ for $t < 0$. The respective corresponding outputs have $y_1(-\frac{1}{2}) = x_1(-\frac{1}{2} - (\frac{1}{8})^2) = \frac{33}{64}$ and $y_1(-\frac{1}{2}) = x_1(-\frac{1}{2} - (\frac{5}{8})^2) = \frac{57}{64}$. These are not equal.

(c) Yes, the system is BIBO stable. For an input $x(t)$ satisfying $\mid x(t) \mid < B$, we have, for all t, that

$$\mid y(t) \mid = \mid x(t - (\int_t^{t+1} x(u) du)^2) \mid < B \ .$$

7. *(7 + 8 points)* Let

$$x_1(t) = \begin{cases} 1 & \text{if } \mid t \mid \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} ,$$

and

$$x_2(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} .$$

(a) Sketch $y(t) = x_1(t) * x_2(t)$. You DO NOT need to write any formulas. The shape of your sketch of $y(t)$ should be accurate and the coordinates should be properly marked.

(b) Let

$$z_1(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 3 & \text{if } 1 < t \leq 2 \\ 0 & \text{otherwise} \end{cases} ,$$

and let

$$z_2(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ \frac{1}{3}t - \frac{1}{3} & \text{if } 1 < t \leq 2 \\ 0 & \text{otherwise} \end{cases} .$$

Let $w(t) = z_1(t) * z_2(t)$. Determine $w(t)$ in terms of $y(t)$ using basic properties of convolution. You need not determine $w(t)$ explicitly : just write it in terms of $y(t)$.

Solution :

(a)

$$y(t) = \begin{cases} \frac{1}{2}(t + \frac{1}{2})^2 & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}(t - \frac{1}{2})^2 & \text{if } \frac{1}{2} \leq t \leq \frac{3}{2} \\ 0 & \text{otherwise} \end{cases} .$$

(b) We have $z_1(t) = x_1(t-\frac{1}{2}) + 3x_1(t-\frac{3}{2})$ and $z_2(t) = x_2(t) + \frac{1}{3}x_2(t-1)$, so

$$w(t) = y(t-\frac{1}{2}) + \frac{10}{3}y(t-\frac{3}{2}) + y(t-\frac{5}{2}) \ .$$

8. *(10 + 10 points)* Consider the function

$$x(t) = \begin{cases} 1 & \text{if } -1 < t \leq 0 \\ 1+t & \text{if } 0 < t \leq 1 \\ 2-t & \text{if } 1 < t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the Fourier transform $X(j\omega)$ of the function $x(t)$.

(b) Let $y(t)$ be the periodic function of period 8 defined by

$$y(t) = \begin{cases} x(t-\frac{3}{2}) & \text{if } 0 < t \leq 4 \\ -x(t+\frac{5}{2}) & \text{if } -4 < t \leq 0 \end{cases} \ .$$

Find the Fourier series coefficients of $y(t)$.

Solution :

(a) Let

$$x_1(t) = \begin{cases} 1 & \text{if } |\, t \,| \leq 1 \\ 0 & \text{otherwise} \end{cases} \ ,$$

and let

$$x_2(t) = \begin{cases} 1 - |\, t \,| & \text{if } |\, t \,| \leq 1 \\ 0 & \text{otherwise} \end{cases} \ .$$

We observe that $x(t) = x_1(t) + x_2(t-1)$. It follows that

$$X(j\omega) = 2\text{sinc}(\frac{\omega}{\pi}) + e^{-j\omega}\text{sinc}^2(\frac{\omega}{2\pi}) \ .$$

(b) Let

$$z(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t-4n) \ .$$

Then

$$y(t) = x(t-\frac{3}{2}) * z(t) \ .$$

It follows that

$$Y(j\omega) = e^{-j\omega\frac{3}{2}} X(j\omega) Z(j\omega) \ .$$

Writing

$$z(t) = \sum_{n=-\infty}^{\infty} \delta(t-8n) - \sum_{n=-\infty}^{\infty} \delta(t-8n-4) \ ,$$

we see that

$$Z(j\omega) = \frac{2\pi}{8}(1 - e^{-j4\omega}) \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{8}k) \ .$$

Hence we have

$$Y(j\omega) = \frac{2\pi}{8}(1 - e^{-j4\omega})e^{-j\omega\frac{3}{2}}X(j\omega) \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{8}k) \ .$$

Now, if $y(t)$ has Fourier series coefficients a_k, we have

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{8}t} \ ,$$

so that

$$Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi}{8}k) \ .$$

Matching coefficients, we see that

$$a_k = \frac{1}{8}(1 - e^{-j4\frac{2\pi}{8}k})e^{-j\frac{2\pi}{8}k\frac{3}{2}}X(j\frac{2\pi}{8}k) \ ,$$

where $X(j\omega)$ is as given above.