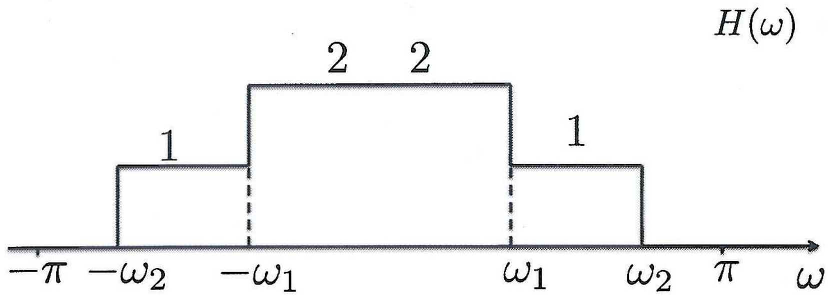
MT2.1 (45 Points) Consider a discrete-time LTI system H whose frequency response $H(\omega)$ is shown below for $-\pi \le \omega \le \pi$.

For every part of this problem, let $\omega_1 = \pi/3$ rad/sec, and $\omega_2 = 2\pi/3$ rad/sec.



(a) (10 Points) Determine a reasonably simple, closed-form expression for $h(n)$, the impulse response of the system.

Method 1 Use DTFT synthesis equation and integrate to find $h(n)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega) e^{i\omega n} d\omega = \frac{1}{2\pi}\left[\int_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} e^{i\omega n} d\omega + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2e^{i\omega n} d\omega + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} e^{i\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{i\omega n}}{in}\Bigg|_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} + \frac{2e^{i\omega n}}{in}\Bigg|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} + \frac{e^{i\omega n}}{in}\Bigg|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right] = \frac{1}{2\pi}\cdot\frac{1}{in}\left(e^{-i\frac{\pi}{3}n} - e^{-i\frac{2\pi}{3}n} + 2e^{i\frac{\pi}{3}n} - 2e^{-i\frac{\pi}{3}n} + e^{i\frac{2\pi}{3}n} - e^{i\frac{\pi}{3}n}\right)$$

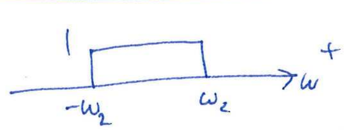
$$= \frac{1}{\pi n}\cdot\frac{1}{2i}\left(e^{i\frac{\pi}{3}n} - e^{-i\frac{\pi}{3}n} + e^{i\frac{2\pi}{3}n} - e^{-i\frac{2\pi}{3}n}\right) = \frac{\sin(\frac{\pi}{3}n) + \sin(\frac{2\pi}{3}n)}{\pi n}$$

(b) (i) (5 Points) Without explicitly carrying out the infinite sum, evaluate $\sum_{n=-\infty}^{+\infty} h(n)$.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n}$$

$$H(0) = \sum_{n=-\infty}^{\infty} h(n) e^{-i0n} = \sum_{n=-\infty}^{\infty} h(n) = 2$$

Method 2 Notice $H(\omega) =$



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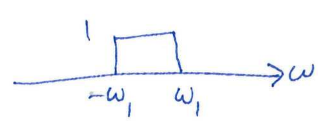


$\updownarrow \mathcal{F}^{-1}$ $\updownarrow \mathcal{F}^{-1}$ From formula sheet pg. 4 $\updownarrow \mathcal{F}^{-1}$

$$h(n) = \frac{\sin(\omega_2 n)}{\pi n} + \frac{\sin(\omega_1 n)}{\pi n}$$

(ii) (5 Points) Without explicitly carrying out the infinite sum, evaluate $\sum_{n=-\infty}^{+\infty} |h(n)|^2$.

By Parseval - Plancherel - Rayleigh identity

$$\sum_{n=-\infty}^{\infty} |h(n)|^2 = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |H(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} 1^2 \, d\omega + \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2^2 \, d\omega + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} 1^2 \, d\omega \right]$$

$$= \frac{1}{2\pi} \left(\frac{\pi}{3} + 4 \cdot \frac{2\pi}{3} + \frac{\pi}{3} \right) = \frac{5}{3}$$

(c) (10 Points) Let x denote the input to the LTI system. H. For each of the following choices of $x(n)$, determine a reasonably simply closed-form expression for the corresponding output $y(n)$.

(i) (5 Points) $x(n) = \cos\left(\frac{\pi n}{6}\right) + \cos\left(\frac{7\pi n}{12}\right) + \sin\left(\frac{3\pi n}{4}\right), \quad \forall n \in \mathbb{Z}.$

$h(n)$ is real-valued $\Rightarrow$

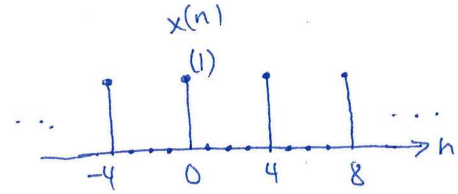
$$\cos(\omega_0 n) \longrightarrow \boxed{H} \longrightarrow |H(\omega_0)| \cos(\omega_0 n + \measuredangle H(\omega_0))$$

$$\sin(\omega_0 n) \longrightarrow \boxed{H} \longrightarrow |H(\omega_0)| \sin(\omega_0 n + \measuredangle H(\omega_0))$$

$$\left.\begin{aligned} H\left(\tfrac{\pi}{6}\right) &= 2 \\ H\left(\tfrac{7\pi}{12}\right) &= 1 \\ H\left(\tfrac{3\pi}{4}\right) &= 0 \end{aligned}\right\}$$

So,

$$y(n) = 2\cos\left(\frac{\pi}{6} n\right) + \cos\left(\frac{7\pi}{12} n\right)$$



(ii) (5 Points) $x(n) = \sum_{\ell=-\infty}^{+\infty} \delta(n - 4\ell), \quad \forall n \in \mathbb{Z}.$

$x(n)$ is periodic with period $p=4$ $\Rightarrow$ DFS $X_k = \frac{1}{4} \sum_{n=-1}^{2} x(n) e^{-ik\frac{2\pi}{4}n} = \frac{1}{4} \quad \forall k$

$\Rightarrow x(n) = \sum_{k=\langle 4 \rangle} \frac{1}{4} e^{-ik\frac{\pi}{2}n}$ and $e^{i\omega_0 n} \rightarrow \boxed{H} \rightarrow H(\omega_0) e^{i\omega_0 n}$

So,

$$y(n) = \sum_{k=\langle 4\rangle} \frac{1}{4} H\left(\tfrac{k\pi}{2}\right) e^{-ik\frac{\pi}{2}n} = \frac{1}{4} \sum_{k=-1}^{2} H\left(\tfrac{k\pi}{2}\right) e^{-ik\frac{\pi}{2}n} = \frac{1}{4} \underbrace{H\left(-\tfrac{\pi}{2}\right)}_{1} e^{i\frac{\pi}{2}n} + \frac{1}{4} \underbrace{H(0)}_{2} + \frac{1}{4} \underbrace{H\left(\tfrac{\pi}{2}\right)}_{1} e^{-i\frac{\pi}{2}n} + \frac{1}{4} \underbrace{H(\pi)}_{0} e^{-i\pi n}$$

$$= \frac{1}{4} e^{i\frac{\pi}{2}n} + \frac{1}{2} + \frac{1}{4} e^{-i\frac{\pi}{2}n} = \underline{\frac{1}{2} + \frac{1}{2}\cos\left(\tfrac{\pi}{2}n\right) = y(n)}$$

(d) (15 Points) In practice, non-causal filters are more difficult to implement and use than causal filters. Suppose we make a causal approximation $\widehat{h}(n)$ to the LTI system by forming $\widehat{h}(n) = h(n)\, u(n)$, where $u(n)$ is the discrete-time unit step. Let the approximation error be $e(n) = h(n) - \widehat{h}(n)$. Evaluate (i.e., find a numerical value for) the energy of the approximation error, which can be expressed by

$$\mathcal{E} = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |H(\omega) - \widehat{H}(\omega)|^2 d\omega.$$

$e(n) = h(n) - \hat{h}(n)$, so $E(\omega) = H(\omega) - \hat{H}(\omega)$

By Parseval-Plancherel-Rayleigh identity:

$$\mathcal{E} = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |E(\omega)|^2 d\omega = \sum_{n=-\infty}^{\infty} |e(n)|^2 = \sum_{n=-\infty}^{\infty} |h(n) - \hat{h}(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |h(n) - h(n)u(n)|^2 = \sum_{n=-\infty}^{-1} |h(n)|^2$$

Note that $h(n)$ is symmetric, i.e. $h(n) = h(-n)$

$$\sum_{n=-\infty}^{\infty} |h(n)|^2 = \sum_{n=-\infty}^{-1} |h(n)|^2 + |h(0)|^2 + \sum_{n=1}^{\infty} |h(n)|^2 = 2 \sum_{n=-\infty}^{-1} |h(n)|^2 + |h(0)|^2$$

equal due to symmetry

So,

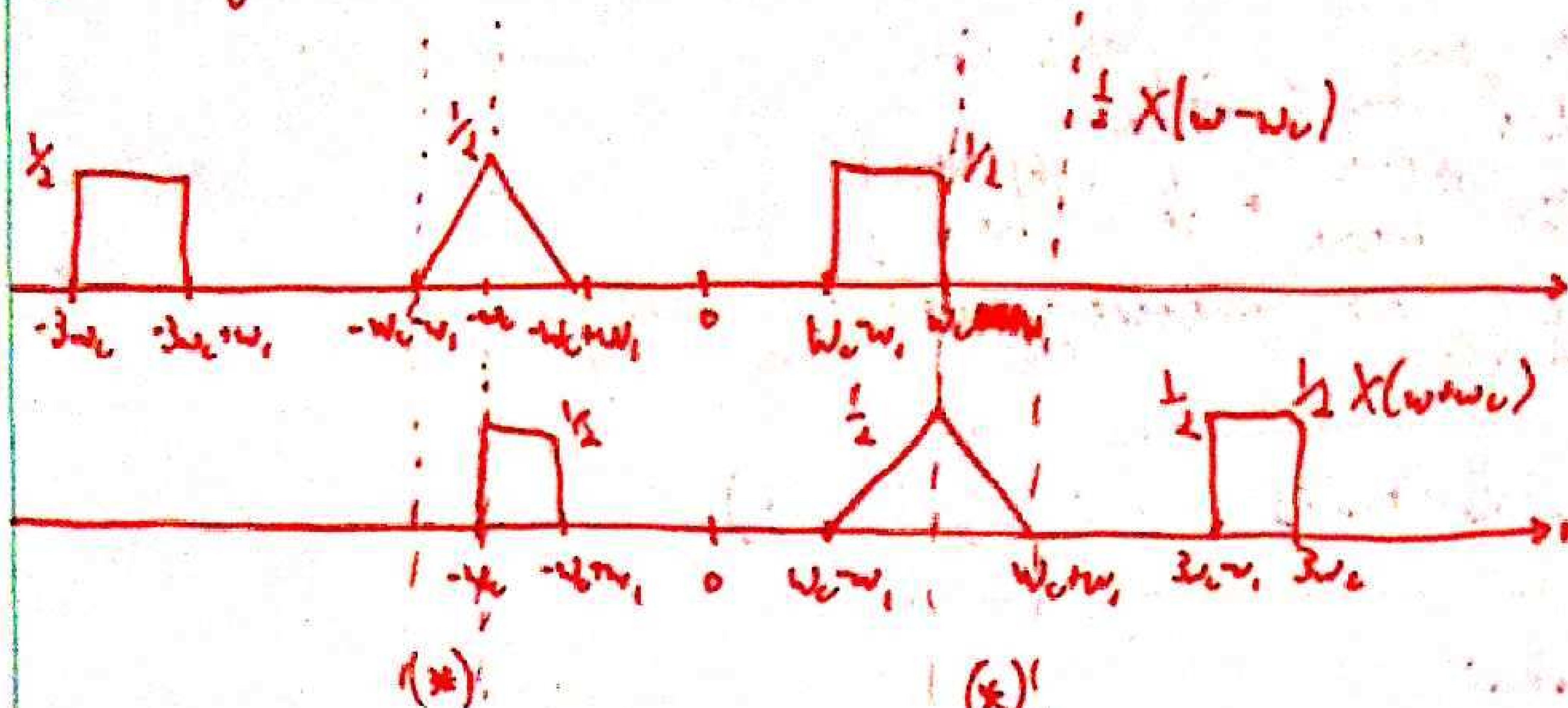
$$\mathcal{E} = \sum_{n=-\infty}^{-1} |h(n)|^2 = \frac{\left(\sum_{n=-\infty}^{\infty} |h(n)|^2\right) - |h(0)|^2}{2} = \frac{\frac{5}{3} - 1}{2} = \underline{\frac{1}{3}}$$

Note: use L'Hopital's rule to find $h(0) = \frac{1}{3} + \frac{2}{3} = 1$

MT2.2 Solution

a) $r(t) = x(t)\cos(\omega_c t) = \frac{1}{2}\left(x(t)e^{i\omega_c t} + x(t)e^{-i\omega_c t}\right)$

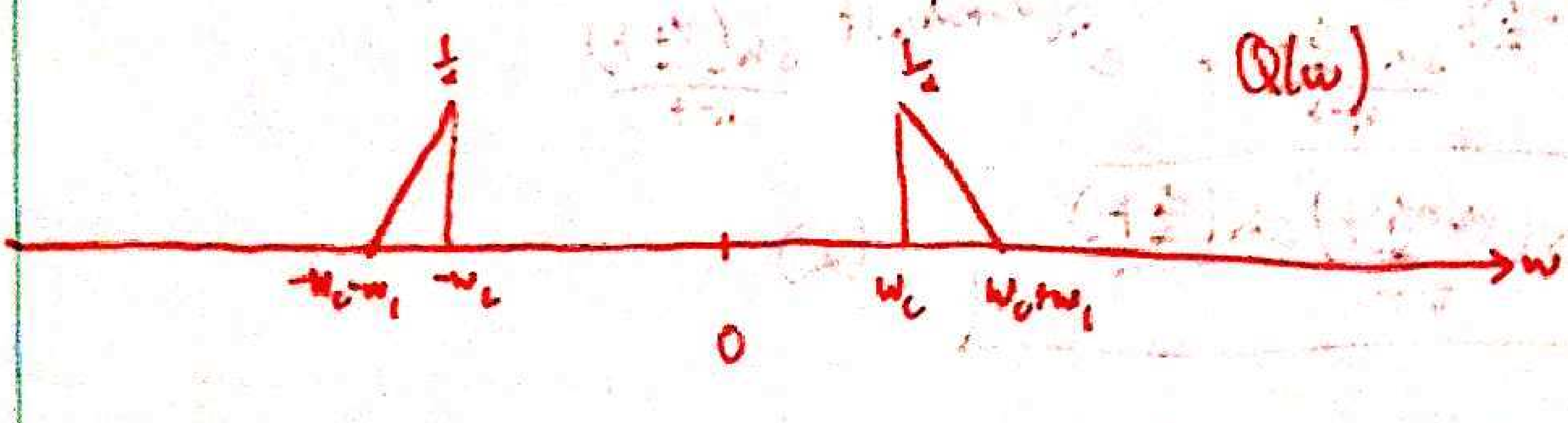
By shifting property and linearity, $R(\omega) = \frac{1}{2}\left(X(\omega+\omega_c) + X(\omega-\omega_c)\right)$.



$Q(\omega) = R(\omega)H(\omega)$, so we only have frequencies $\omega \in [-\omega_c-\omega_1, -\omega_c]$ or $\omega \in [\omega_c, \omega_c+\omega_1]$ in $Q(\omega)$.

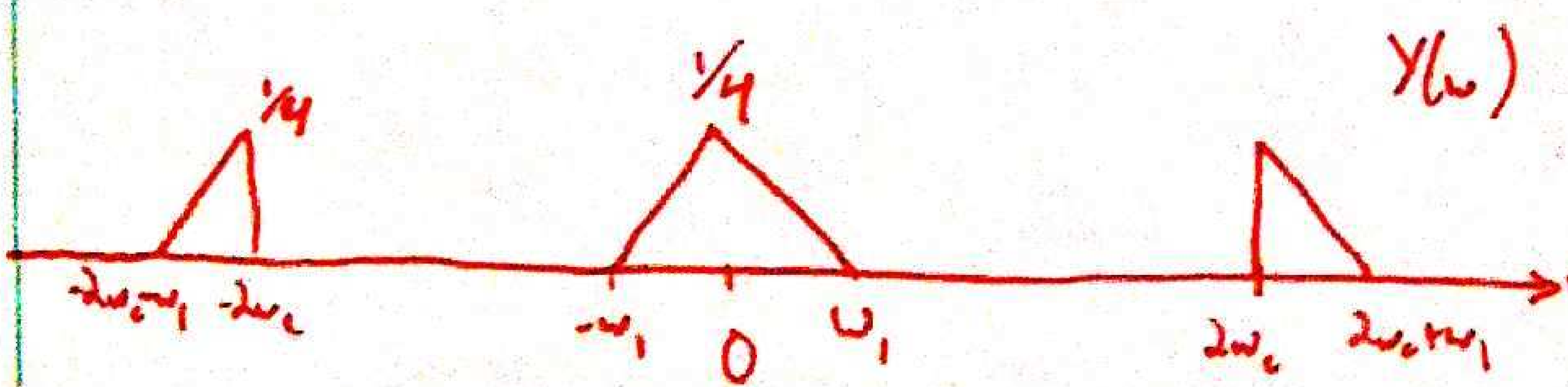
In the above plot, the (*) regions denote the pass-band of the filter H.

Thus,



$y(t) = q(t)\cos(\omega_c t) = \frac{1}{2}\left(q(t)e^{i\omega_c t} + q(t)e^{-i\omega_c t}\right)$

Therefore, $Y(\omega) = \frac{1}{2}\left(Q(\omega+\omega_c) + Q(\omega-\omega_c)\right)$



(b) $h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega$

$$H(\omega) = \begin{cases} 1 & |\omega| \in [\omega_c, \omega_c+\omega_1] \\ 0 & \text{else} \end{cases}$$

$$h(t) = \frac{1}{2\pi}\int_{-\omega_c-\omega_1}^{-\omega_c} e^{i\omega t} d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\omega_c+\omega_1} e^{i\omega t} d\omega$$

$$h(t) = \frac{1}{2\pi}\left[\frac{1}{it} e^{i\omega t}\Big|_{-\omega_c-\omega_1}^{-\omega_c} + \frac{1}{it} e^{i\omega t}\Big|_{\omega_c}^{\omega_c+\omega_1} \right]$$

$$h(t) = \frac{1}{\pi t}\left[\frac{1}{2i}\left(e^{-i\omega_c t} - e^{-i(\omega_c+\omega_1)t} + e^{i(\omega_c+\omega_1)t} - e^{i(\omega_c)t} \right) \right.$$

$$\boxed{h(t) = \frac{1}{\pi t}\left[\sin[(\omega_c+\omega_1)t] - \sin(\omega_c t) \right]} \quad (1)$$

Alternative solution:

Let $H'(\omega) = \begin{cases} 1 & \omega \in [\frac{-\omega_1}{2}, \frac{\omega_1}{2}] \\ 0 & \text{else} \end{cases}$

Observe that $H(\omega) = H'(\omega+\omega_c+\frac{\omega_1}{2}) + H'(\omega-\omega_c-\frac{\omega_1}{2})$.

$$h'(t) = \frac{1}{2\pi}\int_{-\frac{\omega_1}{2}}^{\frac{\omega_1}{2}} e^{i\omega t} d\omega = \frac{1}{2\pi i t}\left(e^{\frac{i\omega_1 t}{2}} - e^{\frac{-i\omega_1 t}{2}}\right) = \frac{\sin(\frac{\omega_1 t}{2})}{\pi t}$$

By shifting property + inverse fourier transform,

$$h(t) = e^{i(\omega_c+\frac{\omega_1}{2})t}\frac{\sin(\frac{\omega_1}{2}t)}{\pi t} + e^{-i(\omega_c+\omega_1/2)t}\frac{\sin(\frac{\omega_1}{2}t)}{\pi t}$$

$$\boxed{h(t) = \frac{2\cos((\omega_c+\frac{\omega_1}{2})t)\sin(\frac{\omega_1}{2}t)}{\pi t}} \quad (2)$$

Note that (1) and (2) are the same, which you can prove with trig. identities.

MT2.3

(a) The problem tells us that G is the set of all real-valued discrete-time signals having a region of support [0,3]. (Note that each of the ψ_k's is in the set G, but they are not the only elements of G!)

To show that G is a subspace of $l^2(\mathbb{Z})$ – the vector-space of all finite-energy discrete-time signals – we first show that G is a subset of $l^2(\mathbb{Z})$. Clearly, this is true, because each signal in G is a discrete-time signal, and each signal x in G has finite energy ($x(0)^2 + x(1)^2 + x(2)^2 + x(3)^2 < \infty$). Now that we have established that G is a subset of $l^2(\mathbb{Z})$, we can prove that G is a subspace by showing the following three things:

(i) G is non-empty.

Proving this is easy – clearly G is not an empty set (for example, the zero-signal is an element of G).

(ii) G is closed under addition.

To prove this, let x and y be two signals in G and let z denote their sum. Then since x and y are real-valued discrete-time signals, z must also be a real-valued and discrete-time signal. Furthermore, if x and y only have support from [0,3], then the same will hold true for z. Thus by definition, z is also in G, and we see that G is closed under addition.

(iii) G is closed under scalar multiplication.

To prove this, let x be a signal in G, and let α be a real-valued scalar. Since x is a real-valued discrete-time signal and α is real-valued, then αx must also be a real-valued and discrete-time signal. Furthermore, if x only has support from [0,3], then the same will hold true for αx. Thus by definition, αx is also in G, and we see that G is closed under addition.

Thus, we have shown that G is a subspace of $l^2(\mathbb{Z})$, and by definition, a subspace is also a vector space.

(b) We need to show three things:

(i) The ψ_k's are all mutually orthogonal.

To prove this, we can compute all of the pairwise inner products, and show that they are equal to 0:

$$\begin{aligned}
\langle \psi_0, \psi_1 \rangle &= \langle \psi_1, \psi_0 \rangle = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot -\frac{1}{2} + \frac{1}{2} \cdot -\frac{1}{2} = 0 \\
\langle \psi_0, \psi_2 \rangle &= \langle \psi_2, \psi_0 \rangle = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot -\frac{1}{\sqrt{2}} = 0 \\
\langle \psi_0, \psi_3 \rangle &= \langle \psi_3, \psi_0 \rangle = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot -\frac{1}{\sqrt{2}} = 0 \\
\langle \psi_1, \psi_2 \rangle &= \langle \psi_2, \psi_1 \rangle = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot -\frac{1}{\sqrt{2}} = 0 \\
\langle \psi_1, \psi_3 \rangle &= \langle \psi_3, \psi_1 \rangle = -\frac{1}{2} \cdot \frac{1}{\sqrt{2}} + -\frac{1}{2} \cdot -\frac{1}{\sqrt{2}} = 0 \\
\langle \psi_2, \psi_3 \rangle &= \langle \psi_3, \psi_2 \rangle = 0
\end{aligned}$$

(ii) The ψ_k's are all "unit length" (normal).

To prove this, we can compute the following inner products, and show that they are equal to 1:

$$\begin{aligned}
||\psi_0|| = \sqrt{\langle \psi_0, \psi_0 \rangle} &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1 \\
||\psi_1|| = \sqrt{\langle \psi_1, \psi_1 \rangle} &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1 \\
||\psi_2|| = \sqrt{\langle \psi_2, \psi_2 \rangle} &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1 \\
||\psi_3|| = \sqrt{\langle \psi_3, \psi_3 \rangle} &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1
\end{aligned}$$

(iii) The ψ_k's form a basis for G.

To prove this, we know from (i) that the ψ_k's are mutually orthogonal – thus they form a set of linearly independent vectors. By inspection, the dimension of G is 4 (it is straightforward to see that a basis for G is $\{\delta[n], \delta[n-1], \delta[n-2], \delta[n-3]\}$, which has 4 elements, and thus the dimension of G 4). Thus, since the ψ_k's are a set of 4 linearly independent vectors in G, they must form a basis for G.

(c) We can write

$$\begin{aligned}
\langle x, \psi_l \rangle &= \langle \sum_{k=0}^{3} X_k \psi_k, \psi_l \rangle \\
&= \sum_{k=0}^{3} X_k \langle \psi_k, \psi_l \rangle \\
&= \sum_{k=0}^{3} X_k \delta(k-l) \\
&= X_l
\end{aligned}$$

Thus we have that

$$\begin{aligned}
X_l &= \langle x, \psi_l \rangle \\
&= \sum_{n=0}^{3} x(n) \psi_l(n)
\end{aligned}$$

Using the values of $x(n)$ and the values of $\psi_l(n)$ for each $l = 0, 1, 2, 3$ in the problem, we can simply plug into the equation above to get

$$\begin{aligned}
X_0 &= \frac{1}{2}(a+b+c+d) \\
X_1 &= \frac{1}{2}(a+b-c-d) \\
X_2 &= \frac{1}{\sqrt{2}}(a-b) \\
X_3 &= \frac{1}{\sqrt{2}}(c-d)
\end{aligned}$$

(d) Thinking geometrically, if we can only use the basis elements ψ_0 and ψ_1, then as discussed in lecture, since the ψ_k's are orthogonal, our best choices of α_0 and α_1 for approximating $\hat{x}$ can be obtained by projecting x onto the subspace H. To do this, we project x onto each of the basis vectors (ψ_0 and ψ_1) of that space. First we calculate the projection of x onto ψ_0:

$$\begin{aligned}
\frac{\langle x, \psi_0 \rangle}{||\psi_0||} &= \langle x, \psi_0 \rangle \\
&= \langle \sum_{k=0}^{3} X_k \psi_k, \psi_0 \rangle \\
&= \sum_{k=0}^{3} X_k \; \langle \psi_k, \psi_0 \rangle \\
&= \sum_{k=0}^{3} X_k \; \delta(k-0) \\
&= X_0
\end{aligned}$$

Thus, we should choose $\alpha_0 = x_0$. A similar projection of x onto ψ_1 will show that $\alpha_1 = X_1$.

Another way to do the problem is to expand out the expression for the approximation error:

$$\begin{aligned}
\epsilon &= \sum_{n=0}^{3} |e(n)|^2 \\
&= \sum_{n=0}^{3} |x(n) - \hat{x}(n)|^2 \\
&= \sum_{n=0}^{3} |\sum_{k=0}^{3} X_k \psi_k(n) - \sum_{k=0}^{1} \alpha_k \psi_k(n)|^2 \\
&= \sum_{n=0}^{3} |(X_0 - \alpha_0)\psi_0(n) + (X_1 - \alpha_1)\psi_1(n) + X_2\psi_2(n) + X_3\psi_3(n)|^2 \\
&= \sum_{n=0}^{3} (X_0 - \alpha_0)^2 \psi_0^2(n) + \sum_{n=0}^{3} (X_1 - \alpha_1)^2 \psi_1^2(n) + \sum_{n=0}^{3} X_2^2 \psi_2^2(n) + \sum_{n=0}^{3} X_3^2 \psi_3^2(n) \\
&= (X_0 - \alpha_0)^2 \sum_{n=0}^{3} \psi_0^2(n) + (X_1 - \alpha_1)^2 \sum_{n=0}^{3} \psi_1^2(n) + X_2^2 \sum_{n=0}^{3} \psi_2^2(n) + X_3^2 \sum_{n=0}^{3} \psi_3^2(n) \\
&= (X_0 - \alpha_0)^2 + (X_1 - \alpha_1)^2 + X_2^2 + X_3^2
\end{aligned}$$

where in the 5th equation above, we omitted cross terms involving different ψ_k's because those terms evaluate to 0 (because the ψ_k's are mutually orthogonal, and so their inner product is zero). From the last line, we immediately see that to minimize the approximation error energy, we should choose $\alpha_0 = X_0$ and $\alpha_1 = X_1$. The corresponding approximation error energy is then $X_2^2 + X_3^2$.

(e) From part (d), we know that the approximation error energy is equal to the sum of the squares of the X_k's corresponding to the basis functions that weren't used in our approximation. Therefore, to minimize the error energy, we should choose basis functions such that the sum of the squares of the corresponding X_k's is maximized. Thus, to inform our choice of basis functions for this part, let us first calculate the X_k's for the signal x that is given. Using our equations from part (c), we get that:

$$\begin{aligned}
X_0 &= 1 \\
X_1 &= 1 \\
X_2 &= \sqrt{2} \\
X_3 &= \sqrt{2}
\end{aligned}$$

Based on our reasoning, from the above numbers, we see that we should pick ψ_2 and ψ_3. The energy of the approximation error signal is then

$$\begin{aligned} \epsilon &= X_0^2 + X_1^2 \\ &= 1^2 + 1^2 \\ &= 2 \end{aligned}$$