EE123 Midterm 1 Solutions

October 22, 2003 — Dan Hazen

- **Problem 1**

(a) We know that $x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$ and $y^*(-n) \overset{\mathcal{F}}{\longleftrightarrow} Y^*(e^{j\omega})$. This is because we know:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} y^*(-n)e^{-j\omega n} &= (\sum_{n=-\infty}^{\infty} y(-n)e^{j\omega n})^* \\
&= (\sum_{k=-\infty}^{\infty} y(k)e^{-j\omega k})^* \\
&= (Y(e^{j\omega}))^* = Y^*(e^{j\omega})
\end{aligned}$$

Finally, we see that the desired signal in the frequency domain is the multiplication of two Fourier transforms, so in the time domain it must be a convolution. We get:

$$z(n) = x(n) \star y^*(-n) = \sum_{k=-\infty}^{\infty} x(k)y^*(k-n) \overset{\mathcal{F}}{\longleftrightarrow} Z(e^{j\omega}) = X(e^{j\omega})Y^*(e^{j\omega})$$

(b) We see that the expression here is a special case of the expression found in part (a). Particularly, we have:

$$z(0) = \sum_{k=-\infty}^{\infty} x(k)y^*(k)$$

but also from the definition of the Fourier transform, we have:

$$z(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})e^{j\omega n}d\omega$$

$$z(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Or putting it all together, we get:

$$\sum_{k=-\infty}^{\infty} x(k)y^*(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

(c) Let $x(n) = sin(\pi n/4)/2\pi n$, $y^*(n) = y(n) = sin(\pi n/6)/5\pi n$, where the equality for $y(n)$ holds because the sinc signal given is purely real. Taking Fourier transforms, we get:

$$X(e^{j\omega}) = \begin{cases} 0.5 & |\omega| < \pi/4 \\ 0 & \text{otherwise} \end{cases}$$

$$Y^*(e^{j\omega}) = Y(e^{j\omega}) = \begin{cases} 0.2 & |\omega| < \pi/6 \\ 0 & \text{otherwise} \end{cases}$$

Note that the above definitions only define one period of the spectrum of $X(e^{j\omega})$ and $Y(e^{j\omega})$. These spectra are periodic with period 2π. Using the result from b, we find:

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi}\int_{\pi/6}^{\pi/6} 0.1 d\omega = \frac{1}{2\pi 10}\int_{\pi/6}^{\pi/6} d\omega = \frac{1}{2\pi 10}\frac{2\pi}{6} = \frac{1}{60}$$

- **Problem 2**

(a) We know that $y(n)$ is stable, so the ROC of $Y(z)$ must conatin the unit circle. Thus, the ROC for $Y(z)$ is $1/2 < |z| < 2$.

(b) Because the ROC is an annulus, $y(n)$ is two-sided.

(c) $x(n)$ is also stable. So, since the ROC must contain the unit circle, we have the ROC of $X(z)$ is $|z| > 3/4$.

(d) One must be careful not to confuse right-sided with causality. By inspection, we know it's right-sided due to the answer for part c. $X(z)$ is causal only if the ROC has no poles at $z = \infty$. This is because if $X(z)$ is causal, it has no terms that are positive powers of z. Thus, since positive powers of z cause poles at $z = \infty$, the ROC cannot have a pole there. Looking at the pole-zero diagram, we see all poles are accounted for within the finite z-plane. So, $x(n)$ is causal.

(e) As mentioned in discussion section, if a pole or zero does not appear on the pole-zero diagram, you can assume it's at infinity. Also, as discussed, we must have an equal number of poles and zeros for any z-transform. Thus, we see that there must be a zero at $z = \infty$. So, since we know $lim_{z \to \infty} X(z) = x(0)$, we have $x(0) = 0$. Note: there are more formal ways of solving this question by looking at the poles and zeros given.

(f) Note that we have pole-zero cancellation. The plot of $H(z)$ is shown in figure 1. The ROC for this system must give an ROC Y that is at least the intersection of the ROC of X and the ROC of H. Thus, the ROC of $H(z)$ is $|z| < 2$.

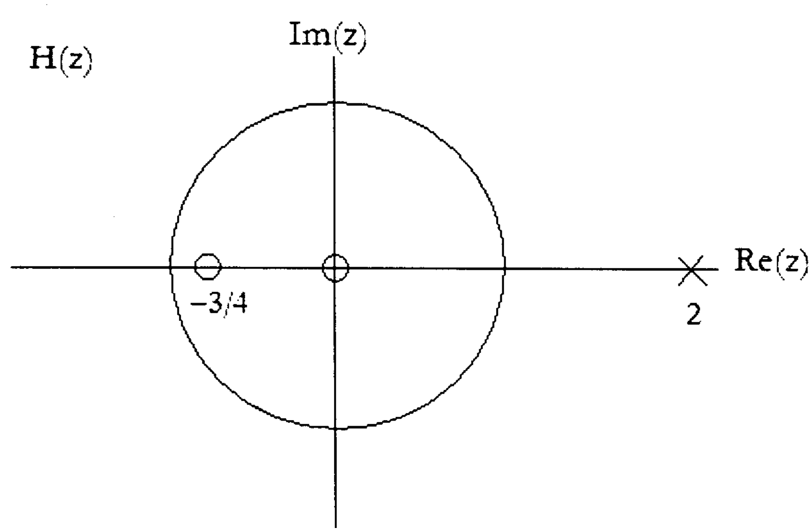


Figure 1: ROC of $H(z)$

(g) This is similar to (d). Again, we can't just say that left-sided sequences are anti-causal. We must see if there's a pole at $z = 0$, if so then the signal is not anti-causal. Since $H(z)$ has a zero at this point, we know that $h(n)$ is anti-causal.

• **Problem 3**

(a) We see that $w(n)$ is just interleaved with the 2 desired signals. Thus, to get $x_1(n)$ back, we need only downsample $w(n)$ to get the even samples. The idea for finding $x_2(n)$ is similar, but we need some kind of delay or time shift. If we try the standard guess of a delay, we get our reconstructed $x_2(n)$, $x_{2Rec}(n) = x_2(n-1)$. This is not the relationship we want, so it can't be a delay. In fact, if we see that $x_2(n) = w(2n+1)$, we see that we must look into the future. Thus, we get that the system looks like that of figure 2.

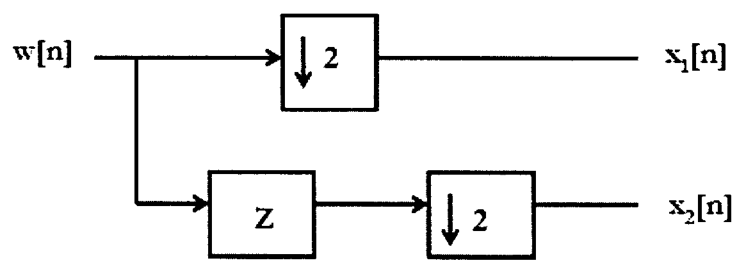
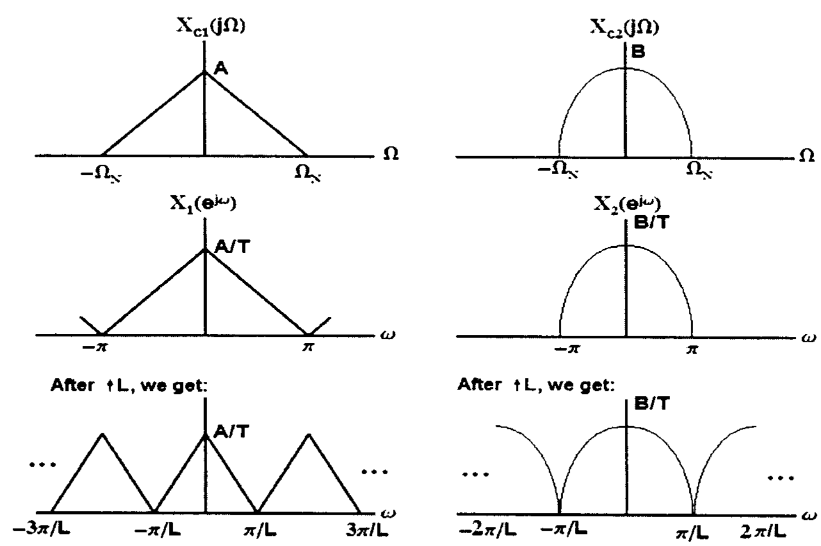
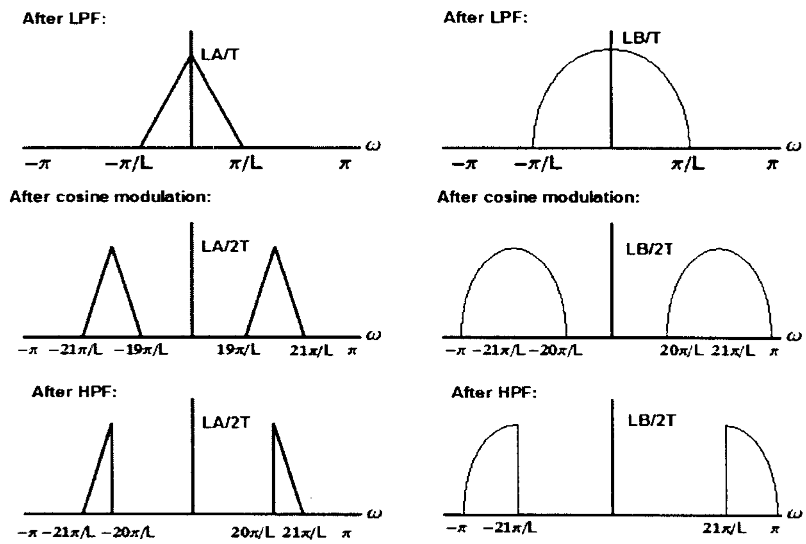


Figure 2: Block Diagram for Question 3 part a

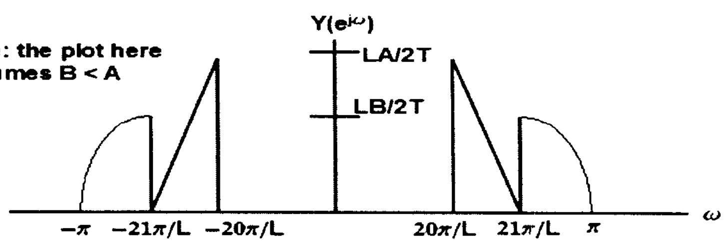
(b) We are given that $T = \frac{\pi}{\Omega_N} = \frac{\pi}{2\pi * 5000} = 10^{-4}$ sec. By sampling, we have $\frac{L\omega_1}{T} = 2\pi * 10^5$ since w_1 is the smallest frequency passed by either HPF (since $w_2 > w_1$). Thus, $w_1 = \frac{20\pi}{L}$. Next, we must find an appropriate L. We know $\omega_2 = \omega_1 + \pi/L$. Also, we know $\omega_2 + \pi/L \leq \pi$ or $L \geq 22$. Selecting the smallest L, we get $L = 22$, $w_1 = \frac{20\pi}{22}$.

(c) Using a generic value of L for our plots, assuming that $B > A$, and realizing that both signals are obtained by sampling at the Nyquist rate, we get the following figures:





Note: the plot here assumes $B < A$



and finally,