Department of EECS - University of California at Berkeley
EECS126 - Probability and Random Processes - Fall 2003
Midterm No. 1: 10/3/2003

SOLUTIONS

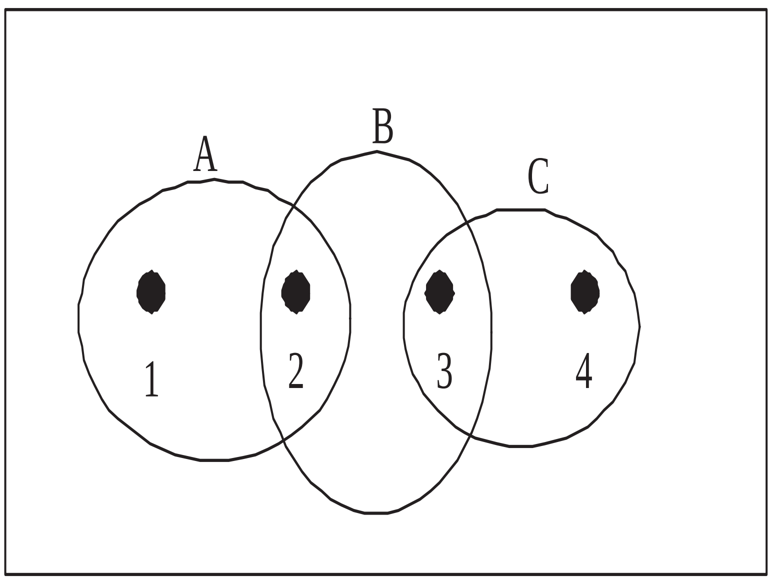
Question 1 (15%). Is it true that

$$P(A \cap B \cap C) = P[A \mid B]P[B \mid C]P(C)?$$

If true, provide a proof; if false, provide a counterexample.

That identity is false. Here is one counterexample. Let $\Omega = \{1, 2, 3, 4\}$ and $p_\omega = 1/4$ for $\omega \in \Omega$. Choose $A = \{1, 2\}, B = \{2, 3\}, C = \{3, 4\}$. Then $P(A \cap B \cap C) = 0, P[A \mid B] = P[B \mid C] = P(C) = 1/2$, so that the identity does not hold.

Here is an illustration of the example.



Question 2 (15%). Describe the probability space $\{\Omega, \mathcal{F}, P\}$ that corresponds to the random experiment "picking five cards without replacement from a perfectly shuffled 52-card deck."

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1. One can choose Ω to be all the permutations of $A := \{1, 2, \ldots, 52\}$. The interpretation of $\omega \in \Omega$ is then the shuffled deck. Each permutation is equally likely, so that $p_\omega = 1/(52!)$ for $\omega \in \Omega$. When we pick the five cards, these cards are $(\omega_1, \omega_2, \ldots, \omega_5)$, the top 5 cards of the deck.

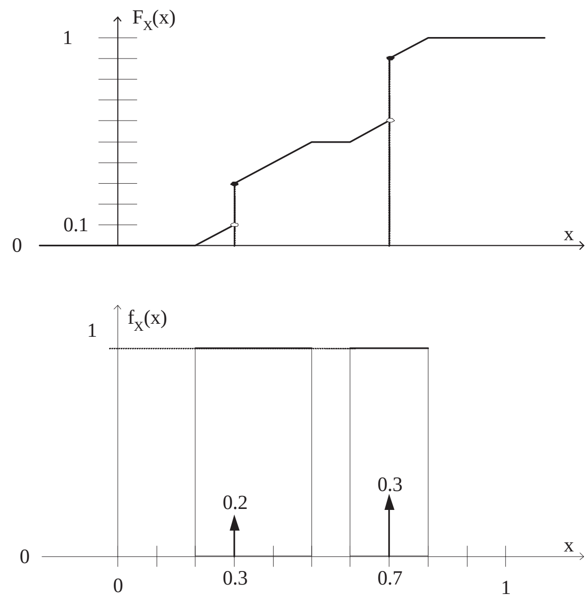
2. One can also choose Ω to be all the subsets of A with five elements. In this case, each subset is equally likely and, since there are $N := \binom{52}{5}$ such subsets, one defines $p_\omega = 1/N$ for $\omega \in \Omega$.

3. One can choose $\Omega = \{\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) \mid \omega_n \in A \text{ and } \omega_m \neq \omega_n, \forall m \neq n, m, n \in \{1, 2, \ldots, 5\}\}$. In this case, the outcome specifies the order in which we pick the cards. Since there are $M := 52!/(47!)$ such ordered lists of five cards without replacement, we define $p_\omega = 1/M$ for $\omega \in \Omega$.

As this example shows, there are multiple ways of describing a random experiment. What matters is that Ω is large enough to specify completely the outcome of the experiment.

Question 3 (20%). Choose X in $[0,1]$ as follows. With probability 0.2, $X = 0.3$; with probability 0.3, $X = 0.7$; otherwise, X is uniformly distributed in $[0.2, 0.5] \cup [0.6, 0.8]$. (a). Plot the c.d.f. of X; (b) Find $E(X)$; (c) Find $var(X)$; (d) Calculate $P[X \leq 0.4 \mid X \geq 0.5]$.

The figure shows the p.d.f. and the c.d.f. of X.



(b) We find

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x dF_x(X) = 0.2 \times 0.3 + 0.3 \times 0.7 + \int_{0.2}^{0.5} x \times 1 dx + \int_{0.6}^{0.8} x \times 1 dx \\
&= 0.27 + [\frac{x^2}{2}]_{0.2}^{0.5} + [\frac{x^2}{2}]_{0.6}^{0.8} = 0.27 + \frac{1}{2}(0.25 - 0.04 + 0.64 - 0.36) = 0.515.
\end{aligned}
$$

(c) We first compute $E(X^2)$. We get

$$
\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} x^2 dF_x(X) = 0.2 \times (0.3)^2 + 0.3 \times (0.7)^2 + \int_{0.2}^{0.5} x^2 \times 1 dx + \int_{0.6}^{0.58} x^2 \times 1 dx \\
&= 0.165 + [\frac{x^3}{3}]_{0.2}^{0.5} + [\frac{x^3}{3}]_{0.6}^{0.8} = 0.165 + \frac{1}{3}(0.125 - 0.008 + 0.512 - 0.216) = 0.3027.
\end{aligned}
$$

Hence,

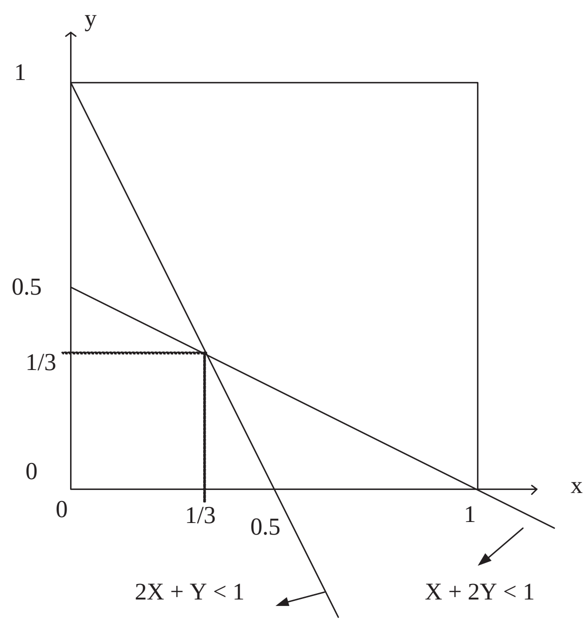
$$
var(X) = E(X^2) - (E(X))^2 = 0.3027 - (0.515)^2 = 0.0374.
$$

(d) Finally,

$$
P[X \leq 0.4 \mid X \geq 0.5]] = \frac{P(X \leq 0.4)}{P(X \leq 0.5)} = \frac{0.2 + 0.2}{0.2 + 0.3} = 0.8.
$$

Question 4 (15%). Let (X, Y) be the coordinates of a point picked randomly and uniformly in $[0,1]^2$. Calculate $P[X + 2Y \leq 1 \mid 2X + Y \leq 1]$.

The figure below shows the relevant sets of outcomes.



From the figure we see that

$$P(X + 2Y \leq 1 \text{ and } 2X + Y \leq 1) = \frac{1}{3} \times \frac{1}{3} + (\frac{1}{2} - \frac{1}{3}) \times \frac{1}{3} = \frac{1}{6}.$$

Also,

$$P(2X + Y \leq 1) = \frac{1}{2} \times (\frac{1}{2} \times 1) = \frac{1}{4}.$$

Hence,

$$P[X + 2Y \leq 1 \mid 2X + Y \leq 1] = \frac{1/6}{1/4} = \frac{2}{3}.$$

Question 5 (15%). Let X be a random variable that is exponentially distributed with mean 1. Calculate $P[X \in [1,4] \mid X \in [3,5]]$.

This is straightforward if we recall that $F_X(x) = 1 - e^{-x}$ for $x \geq 0$ and $F_X(x) = 0$ for $x \leq 0$. One has

$$P[X \in [1,4] \mid X \in [3,5]] = \frac{P(X \in [3,4])}{P(X \in [3,5])}.$$

Now,

$$P(X \in [3,4]) = F_X(4) - F_X(3) = (1 - e^{-4}) - (1 - e^{-3}) = e^{-3} - e^{-4}.$$

Similarly,

$$P(X \in [3,5]) = e^{-3} - e^{-5}.$$

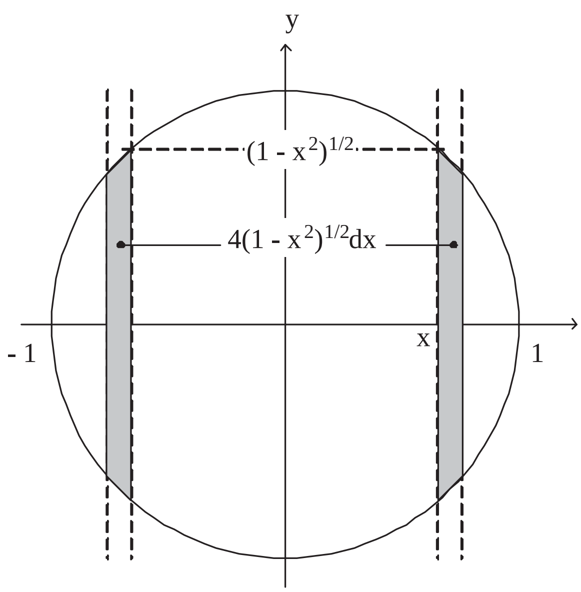
Hence,

$$P[X \in [1,4] \mid X \in [3,5]] = \frac{e^{-3} - e^{-4}}{e^{-3} - e^{-5}} \approx 0.73.$$

Question 6 (20%). Let (X, Y) be the coordinates of a point picked uniformly in $\{(x, y) \in \Re^2 \mid x^2 + y^2 \leq 1\}$. Calculate $E(|X|)$.

(*Hint:* First find f_Y where $Y = |X|$. To do that, look at the set of outcomes such that $Y \in (x, x + dx)$ and determine its probability.)

The figure below shows the set of points (X, Y) with $X \in (x, x + dx)$ for $0 \leq x < 1$.



As the figure shows, the area of that set is $4\sqrt{1 - x^2}dx$. Since the area of the circle is π, this implies that (with $Y = |X|$)

$$f_Y(x) = \frac{4}{\pi}\sqrt{1 - x^2}, \text{ for } x \in [0, 1].$$

Consequently,

$$E(Y) = \int_0^1 x\frac{4}{\pi}\sqrt{1 - x^2}dx.$$

Note that the derivative of $(1 - x^2)^{3/2}$ is

$$\frac{3}{2}(1 - x^2)^{1/2} \times (-2x) = -3\sqrt{1 - x^2}.$$

Hence,

$$E(Y) = -\int_0^1 \frac{4}{\pi}\frac{1}{3}d[(1 - x^2)^{3/2}] = -\frac{4}{3\pi}[(1 - x^2)^{3/2}]_0^1 = \frac{4}{3\pi} \approx 0.42.$$