EE126: Probability and Random Processes **F'10**

Midterm — October 26

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Formulas: Given the short attention span induced by twitter and the like, we thought you might appreciate not having to remember the following formulas. After all, they are on Wikipedia.

$$\mathbf{X} = N(\mu, \Sigma) \Leftrightarrow f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\}$$

$$(\mathbf{X}, \mathbf{Y}) J.G. \Rightarrow E[\mathbf{X}|\mathbf{Y}] = E(\mathbf{X}) + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{Y}}^{-1}(\mathbf{Y} - E(\mathbf{Y}))$$

$$\text{cov}(\mathbf{AX}, \mathbf{BY}) = \mathbf{A}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}^T.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $V = N(0,1)$, then $P(V > 1) = 0.159, P(V > 1.64) = 0.05,$
$P(V > 1.96) = 0.025, P(V > 2) = 0.023, P(V > 2.58) = 0.005.$

Problem 1. *(***Short Problems 40%***)*

- *Give an example where $E[X|Y] = E(X)$ but X, Y are not independent.*

- *Let X, Y be i.i.d., $B(100, 0.3)$. Calculate $E[X - Y|X + Y]$.*

- *Let X, Y be as in the previous problem. Calculate $E[(X + Y)^2|X]$.*

- *Assume that* $\Sigma_X = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. *Calculate* $\text{var}(2X_1 + 3X_2 + X_3)$.

- *Let* X, Y, Z *be i.i.d.* $U[0,1]$. *Calculate* $E[2X + 3Y + 4Z | X + Y + Z]$.

Problem 2. *(**20%**) Assume that $Y = X + aZ$ where X, Z are i.i.d., N(0, 1).*

(a) Calculate $E[X|Y]$;

(b) What is the variance of X given Y?

(c) Given $Y = y$, what is the distribution of X?

(d) Express $P[X > c|Y = y]$ in terms of $\Phi(z) := P(Z \leq z)$ where $Z = N(0,1)$.

Problem 3. *(**20%**) Assume that* $Y = 4X + Z$ *where* $Z = N(0, 1)$ *and* $P(X = k) = 1/3$ *for* $k \in \{-1, 0, 1\}$.

a) *Calculate* $\hat{X} = MAP[X|Y]$.

b) *Calculate* $P(\hat{X} \neq X)$.

Problem 4. *(**20%**) When $X = 0$, $Y = N(0, 1)$. When $X = 1$, $Y = N(0, 4)$.*

(a) Find $\hat{X} = g(Y)$ that maximizes $P[\hat{X} = 1|X = 1]$ subject to $P[\hat{X} = 1|X = 0] \leq 5\%$.

(b) What is $P[\hat{X} = 1|X = 1]$?