Department of EECS - University of California at Berkeley
EECS126 - Probability and Random Processes - Spring 2001
Midterm No. 2: 4/6/2001

Student Name / SID / email:

Part 1 - Multiple Choice (30%) Each question below counts for 5%.

Problem 1: Is it true that

$$var(X+Y) \geq var(X) + var(Y).$$

NO: Consider $X = -Y$ where $E[X] = 0$ and $var(X) > 0$.

Problem 2: Is it true that

$$E[X|X+Y] = X + E[X|Y].$$

NO: Consider X and Y i.i.d. Uniform$[0,1]$. By symmetry and linearity of conditional expectation

$$E[X \mid X+Y] = \frac{X+Y}{2} \tag{1}$$

but

$$X + E[X \mid Y] = X + \frac{1}{2} \tag{2}$$

Problem 3: Let X, Y, Z be i.i.d. N(0, 1). Is the following formula correct?

$$E[2X+Y-Z|X-Y+Z] = 0.$$

YES: Note that $2X+Y-Z$ and $X-Y+Z$ are Gaussian random variables. Furthermore if they are uncorrelated, they are independent.

$$\begin{aligned} cov(2X+Y-Z, X-Y+Z) &= 2E[X^2] - E[Y^2] - E[Z^2] \\ &= 2-1-1 \\ &= 0 \end{aligned} \tag{3}$$

Since they are uncorrelated, they are independent. Hence

$$\begin{aligned} E[2X+Y-Z|X-Y+Z] &= E[2X+Y-Z] \\ &= 0 \end{aligned} \tag{4}$$

Problem 4: Assume that X and Y are two random variables such that $X+Y$ and $X-Y$ are independent. Is it always true that X and Y are independent?

NO: Consider $X = 1$ and $Y = X$.

$$\begin{aligned} F_{X+Y,X-Y}(u,v) &= P(2 \le u, 0 \le v) \\ &= 1(2 \le u)1(0 \le v) \\ &= F_{X+Y}(u)F_{X-Y}(v) \end{aligned} \tag{5}$$

Problem 5: Is it always true that

$$var(\frac{X+Y}{2}) \le \max\{var(X), var(Y)\}.$$

YES:

$$\begin{aligned} var(\frac{X+Y}{2}) &= \frac{1}{4}(var(X) + var(Y) + 2cov(X,Y)) \\ &\le \frac{1}{4}(2\max(var(X), var(Y)) + 2cov(X,Y)) \end{aligned} \tag{6}$$

But

$$\begin{aligned} E[(X-E[X])(Y-E[Y])]^2 &\le E[(X-E[X])^2]E[(Y-E[Y])^2] \\ &\le \max(var(X), var(Y))^2 \end{aligned} \tag{7}$$

Thus

$$\begin{aligned} cov(X,Y) &= E[(X-E[X])(Y-E[Y])] \\ &\le \max(var(X), var(Y)) \end{aligned} \tag{8}$$

Finally

$$\begin{aligned} var(\frac{X+Y}{2}) &\le \frac{1}{4}(2\max(var(X), var(Y)) + 2cov(X,Y)) \\ &\le \max(var(X), var(Y)) \end{aligned} \tag{9}$$

Problem 6: Let (X, Y) be picked uniformly in a unit circle. Is it true that X and Y are uncorrelated?

YES:

$$\begin{aligned} E[X \mid Y] &= 0 \\ YE[X \mid Y] &= 0 \\ E[YE[X \mid Y]] &= 0 \\ E[E[YX \mid Y]] &= 0 \\ E[YX] &= 0 \\ E[YX] - E[Y]E[X] &= -E[Y]E[X] \\ cov(Y,X) &= 0 \end{aligned} \tag{10}$$

Part 2 - Problem A (20%)

Let $X \in \{0, 1\}$ and Z be N(0, 1). Let also $Y = X + (X+1)Z$. Find the MLE of X given Y.

SOLUTION:

Conditioned on $\{X = 0\}$, $Y \sim N(0, 1)$, and conditioned on $\{X = 1\}$, $Y \sim N(1, 4)$.

$$
\begin{aligned}
f_{Y|X=1}(Y \mid X = 1) &\underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} f_{Y|X=0}(Y \mid X = 0) \\
\frac{1}{\sqrt{8\pi}} e^{-\frac{1}{8}(Y-1)^2} &\underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Y^2} \\
e^{-\frac{1}{8}(Y-1)^2} &\underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} 2e^{-\frac{1}{2}Y^2} \\
\exp(-\frac{Y^2}{8} + \frac{Y}{4} - \frac{1}{8} + \frac{Y^2}{2} &\underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} 2 \\
2Y^2 + 2Y - 1 &\underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} 8 \log 2 \\
2Y^2 + 2Y - (1 + 8 \log 2) &\underset{\hat{X}=0}{\overset{\hat{X}=1}{\gtrless}} 0
\end{aligned}
\tag{11}
$$

Thus,

$$
\hat{X} = 1(Y \notin [y_0, y_1]) \tag{12}
$$

where

$$
\begin{aligned}
y_0 &= \frac{1 - 1\sqrt{1 + 3(1 + \log 2)}}{3} \\
y_1 &= \frac{1 + 1\sqrt{1 + 3(1 + \log 2)}}{3}
\end{aligned}
\tag{13}
$$

Part 3 - Problem B (25%)

Let (X, Y) be picked uniformly in $[0, 1]^2$. Calculate $L[X|(X + Y)^2]$.

SOLUTION:

$$\begin{aligned} E[(X+Y)^2] &= E[X^2 + 2XY + Y^2] \\ &= \frac{1}{3} + (2)(\frac{1}{2})(\frac{1}{2}) + \frac{1}{3} \\ &= \frac{7}{6} \end{aligned} \tag{14}$$

$$\begin{aligned} var(X+Y)^2 &= E[(X+Y)^4] - (E[(X+Y)^2])^2 \\ &= E[X^4 + 4X^3Y + 6X^2Y^2 + 4XY^3 + Y^4] - (\frac{7}{6})^2 \\ &= \frac{1}{5} + (4)(\frac{1}{4})(\frac{1}{2}) + (6)(\frac{1}{3})(\frac{1}{3}) + (4)(\frac{1}{2})(\frac{1}{4}) + \frac{1}{5} - (\frac{7}{6})^2 \\ &= \frac{31}{15} - \frac{49}{36} \\ &= \frac{127}{180} \end{aligned} \tag{15}$$

$$\begin{aligned} cov(X, (X+Y)^2) &= E[X^3 + 2X^2Y + XY^2] - E[X]E[X^2 + 2XY + Y^2] \\ &= \frac{1}{4} + (2)(\frac{1}{3})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{3}) - (\frac{1}{2})(\frac{1}{3} + (2)(\frac{1}{2})(\frac{1}{2}) + \frac{1}{3}) \\ &= \frac{1}{2} \end{aligned} \tag{16}$$

$$\begin{aligned} L[X \mid (X+Y)^2] &= E[X] + \frac{cov(X, (X+Y)^2)}{var(X+Y)^2}((X+Y)^2 - E[(X+Y)^2] \\ &= \frac{1}{2} + \frac{90}{127}((X+Y)^2 - \frac{7}{6}) \end{aligned} \tag{17}$$

Part 4 - Problem C (25%)

Given $X \in \{0, 1, 2, \ldots\}$, Y is picked uniformly in $\{0, 1, 2, \ldots, X\}$. Assume that $P(X = n) = (n+1)p^n(1-p)^2$ for $n \geq 0$ where p is a known number in $(0, 1)$. Calculate $E[X|Y]$.

SOLUTION:

We need to determine the conditional probability $P(X = n \mid Y = m)$ and then determine the average value of X with respect to this mass function.

$$P(X = n \mid Y = m) = \frac{P(Y = m \mid X = n)P(X = n)}{P(Y = m)} \tag{18}$$

We are given that

$$P(Y = m \mid X = n) = \frac{1}{n+1} 1(0 \leq m \leq n) \tag{19}$$

We obtain $P(Y = m)$ by marginalizing X out of the joint mass function.

$$\begin{aligned} P(Y = m) &= \sum_{n=0}^{\infty} P(X = n, Y = m) \\ &= \sum_{n=0}^{\infty} P(Y = m \mid X = n)P(X = n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} 1(0 \leq m \leq n)(n+1)p^n(1-p)^2 \\ &= \sum_{n=m}^{\infty} p^n(1-p)^2 \\ &= \frac{p^m}{1-p} \end{aligned} \tag{20}$$

Thus

$$\begin{aligned} P(X = n \mid Y = m) &= \frac{\frac{1}{n+1} 1(0 \leq m \leq n)(n+1)p^n(1-p)^2}{\frac{p^m}{1-p}} \\ &= \frac{(1-p)^3}{p^m} 1(0 \leq m \leq n)p^n \end{aligned} \tag{21}$$

Averaging X against this mass function

$$
\begin{aligned}
E[X \mid Y = m] &= \sum_{n=0}^{\infty} nP(X = n \mid Y = m) \\
&= \sum_{n=0}^{\infty} n \frac{(1-p)^3}{p^m} 1(0 \leq m \leq n) p^n \\
&= \frac{(1-p)^3}{p^m} \sum_{n=m}^{\infty} np^n \\
&= \frac{(1-p)^3}{p^m} \sum_{n=m}^{\infty} p \frac{\partial}{\partial p} p^n \\
&= \frac{(1-p)^3}{p^m} p \frac{\partial}{\partial p} \sum_{n=m}^{\infty} p^n \\
&= \frac{(1-p)^3}{p^m} p \frac{\partial}{\partial p} \frac{p^m}{1-p} \\
&= \frac{(1-p)^3}{p^m} \frac{m(1-p)p^m + p^{m+1}}{(1-p)^2} \\
&= m(1-p)^2 + p(1-p)
\end{aligned} \tag{22}
$$

So

$$
E[X \mid Y] = (1-p)^2 Y + p(1-p) \tag{23}
$$