Department of EECS - University of California at Berkeley
EECS126 - Probability and Random Processes - Spring 2003
Midterm No. 1 bis: 4/16/2003

SOLUTIONS

There are four short questions and six slightly longer problems. Answer on these sheets. Show your work. Good luck.

Question 1 (6%). Over a large number of experiments, you watch the number of photons that hit a photodetector in one hour. You record the fraction of experiments when the number of photons is even and when it is odd. You find that the probability that the number is even is 0.6.

a. Describe the probability space $\{\Omega, \mathcal{F}, P\}$ that models that information.

b. Give an example of a function $X : \Omega \to \Re$ that is not a random variable for that probability space.

c. Give a different probability space with the same Ω such that the same function $X(\omega)$ is a random variable on that space.

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a. $\Omega = \{0, 1, 2, 3, 4, 5, \ldots\}.\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$ where $A = \{0, 2, 4, 6, \ldots\}, P(A) = 0.6, P(A^c) = 0.4, P(\emptyset) = 0, P(\Omega) = 1$.

b. $X(\omega) = \omega$.

c. Let $\mathcal{F} = 2^{\Omega}$ and $p_n = \frac{\lambda^n}{n!}\exp\{-\lambda\}$ for $n \geq 0$, $P(A) = \sum_{n \in A} p_n$ for $A \in \mathcal{F}$.

Question 2 (6%). Let $\{\Omega, \mathcal{F}, P\}$ be the probability space that corresponds to rolling a balanced die. Give an example of two events A and B in that probability space that are disjoint and independent.

Let $A = B = \emptyset$. Then $P(A \cap B) = 0 = P(A)P(B)$.

Question 3 (8%). Let X be a Poisson random variable with mean 1. Calculate $P(X \text{ is even})$.

$$P(X \text{ is even}) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \exp\{-1\}.$$

Also,

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \frac{1}{2}\{\sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\} = \frac{1}{2}\{e^1 + e^{-1}\}.$$

Hence,

$$P(X \text{ is even}) = \frac{1}{2}\{1 + e^{-2}\}.$$

Question 4 (6%). Assume that A, B, C are events in $\{\Omega, \mathcal{F}, P\}$ that are mutually independent. Prove that

$$P[A|B^c \cap C] = P(A).$$

By definition of mutual independence, $P(A \cap B \cap C) = P(A)P(B)P(C)$ and $P(A \cap C) = P(A)P(C)$. Subtracting the first identity from the second gives $P(A \cap B^c \cap C) = P(A)P(B^c)P(C)$. Similarly, we find that $P(B^c \cap C) = P(B^c)P(C)$. Now we calculate

$$P[A|B^c \cap C] = \frac{P(A \cap B^c \cap C)}{P(B^c \cap C)} = \frac{P(A)P(B^c)P(C)}{P(B^c)P(C)} = P(A).$$

Problem 1 (8%). You roll a balanced die 4 times. Let X be the sum of the faces of the first and second tosses, Y be the sum of the faces of the second and third tosses, and Z be the sum of the third and fourth tosses. Calculate $E(\max\{X, Y, Z\})$.

The details are long. The key step is to condition on the faces of the second and third die.

Let D_k be the face of die k, for $k = 1, 2, 3, 4$. Given $\{D_2 = m, D_3 = n\}$ we see that $V = \max\{X, Y, Z\} = m + n + \max\{D_1 - n, D_4 - m, 0\}$. Define $f(a, b) = E(\max\{W - b, a\})$ where W is uniformly distributed in $\{1, 2, \ldots, 6\}$ and a, b are integers. We worry about calculating $f(.)$ later.

Then, $E[V|D_2 = m, D_3 = n] = m + n + \frac{1}{6}\sum_{k=1}^{6} f(m, k - n)$. We then can calculate $E(V) = E(E[V|D_2 = m, D_3 = n])$.

To calculate $f(.)$, note that $W - b$ is uniformly distributed in $\{1 - b, 2 - b, \ldots, 6 - b\}$. If a is an integer in that set, then $\max\{W - b, a\}$ is equal to a with probability $(a + b)/6$ and is uniformly distributed in $\{a, a + 1, \ldots, 6 - b\}$ otherwise. Hence,

$$f(a, b) = \frac{a(a + b) + (6 - a - b)(7 + a - b)}{6}.$$

Putting the pieces together allows to calculate $E(V)$.

Problem 2 (6%). Let X and Y be two independent random variables that are exponentially distributed with mean 1. Calculate $E(\max\{X, 2Y\})$.

We could do this brute force. Instead, let's use what we know about exponential random variables. The random variable $2Y$ is exponentially distributed with rate $1/2$. We then have to compute the mean value of the maximum of two independent exponentially distributed random variables, X with rate $\lambda = 1$ and Y with rate $\mu = 1/2$.

Note that

$$\max\{X, Y\} = \min\{X, Y\} + (\max\{X, Y\} - \min\{X, Y\}).$$

Also, given $\min\{X, Y\}$, the random variable $\max\{X, Y\} - \min\{X, Y\}$ is the residual value of X if $Y < X$ and the residual value of Y if $X < Y$. By the memoryless property of the exponential distribution, we see that

$$E(\max\{X, Y\}) = (\lambda + \mu)^{-1} + \frac{\lambda}{\lambda + \mu}\frac{1}{\mu} + \frac{\mu}{\lambda + \mu}\frac{1}{\lambda}.$$

Indeed, $P[X < Y \mid \min\{X, Y\}] = \frac{\lambda}{\lambda+\mu}$.

Problem 3 (10%). Let X, Y be independent and uniformly distributed in $[0, 1]$. Find $E(\min\{2X - Y, X + Y\})$.

Note that $2X - Y \leq X + Y$ if and only if $X \leq 2Y$. Hence the required expected value is

$$\int_0^1 [\int_0^{x/2} (x+y)dy + \int_{x/2}^1 (2x-y)dy]dx.$$

The calculation is straightforward.

Problem 4 (8%). You pick a point in the unit circle with the uniform distribution. Designate the Cartesian coordinates of the point by (X, Y). Find $P(X > 3Y)$.

This is elementary (draw a picture of the unit square ...).

Problem 5 (12%). You pick a point X in the unit interval $[0, 1]$ with the uniform distribution. Plot the c.d.f. of $Y = \max\{0.2, |X - 0.4|\}$. Calculate $E(Y)$ and $var(Y)$.

Draw a plot of Y as a function of X. You see that $0.2 \leq Y \leq 0.6$. Also, for $y \in [0.4, 0.6], P(Y > y) = 0.6 - y$. For $y \in (0.2, 0.4), P(Y > y) = 0.2 + 2(0.4 - y)$. Putting these facts together, you can plot the c.d.f. of Y and calculate the mean and variance.

Problem 6 (10%). There are two coins. One coin is fair, the other is biased with $P(H) = 0.6$. You are given one of the two coins, the fair one with probability 0.7 and the biased coin with probability 0.3. What is the probability that you got the fair coin given that after tossing it you get 'H'?

This is elementary (Bayes' rule).