EECS 16B Designing Information Devices and Systems II
Spring 2016 Anant Sahai and Michel Maharbiz **Final Exam**

Exam: 1 Pimentel (SIDs ending in 1,4,5,7,8,9)

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ______________, ______________ ________________
(last) (first) (signature)

PRINT your Unix account login: ee16b-__________

PRINT your discussion section and GSI (the one you attend): ________________________________

Row Number (front row is 1): ________ Seat Number (left most is 1): ________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

Section 0: Pre-exam questions (3 points)

1. **What has been the most interesting concept or useful skill you learned from EE16B? (1 pt)**

2. **What are you most looking forward to for the Summer? (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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Section 1: Straightforward questions (80 + 10 pts)

Unless told otherwise, you must show work to get credit. There will be less partial credit given in this section. There are 90 points available, but we consider 10 of them as bonus. So skip ahead instead of getting stuck.

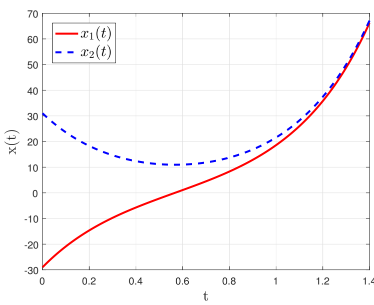
3. Matching Question (9 pts)

Consider the 2-dimensional differential equation

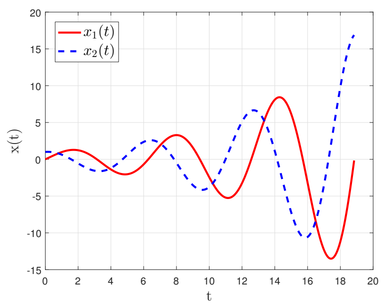
$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

where A is a 2 by 2 real matrix.

With some initial values, solutions as a function of time have been plotted. **Please select the corresponding solution plot for each A matrix.** (No need to show work)



(A)



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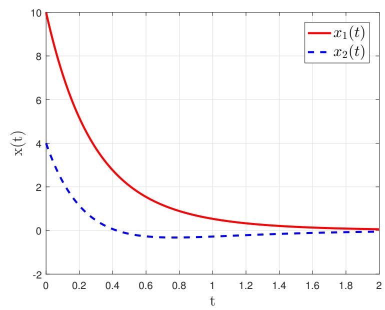

(C)

(a) $A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}$ This has eigenvalues $-4, -2$.

(b) $A = \begin{bmatrix} 0.15 & 1 \\ -1 & 0.15 \end{bmatrix}$ This has eigenvalues $0.15 \pm i$.

(c) $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ This has eigenvalues ± 3.

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4. What to do? (10pts)

You have learned lots of techniques/concepts in this course. These include:

(a) DFT

(b) SVD

(c) K-means

(d) Eigenvalue placement by means of controllable canonical form

For each of the scenarios below, **please choose which technique/concept you should invoke.** No explanations are needed.

(a) You want to design a feedback law to regulate a system to make it stable.

(b) You have a data set and you want to find a lower-dimensional subspace that you can project it onto while still keeping most of the important variation in the data set.

(c) You have some data on how much different individuals like different kinds of food and you want to classify people into several groups based on their preferences.

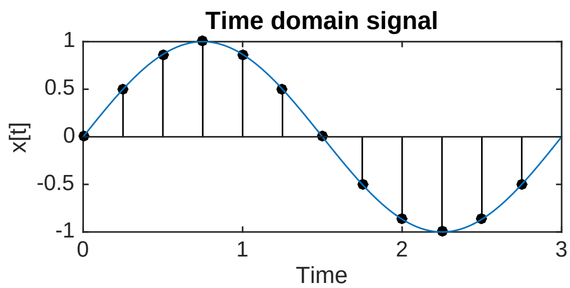
(d) You want to find a control sequence $u(t)$ of length n that will take the state of a linear system from $\vec{0}$ to a desired destination $\vec{p}$ such that the control sequence u viewed as an n-length vector $\vec{u}$ has the smallest Euclidean norm.

(e) You have some measurements at regular intervals from a slowly varying physical process and would like to reconstruct hypothetical measurements for times in between your measurements.

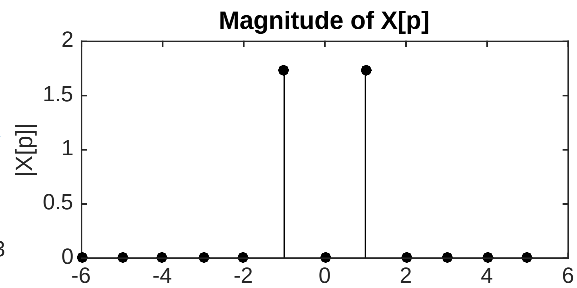
5. Matching: DFT and Sampling (12 pts)

Circle your answer. There is no need to give any justification.

(a) Given the time domain signal below with 12 samples taken over 3 seconds:

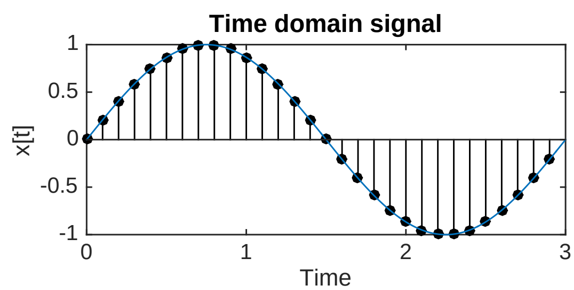


Time domain signal.

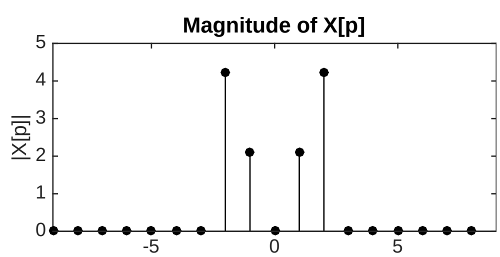


DFT magnitudes.

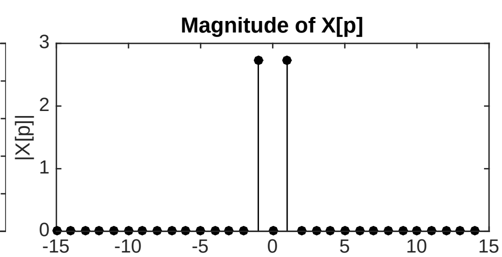
Now, we have sampled the same signal at a different sampling rate to get 30 samples over 3 seconds:



which plot below represents all the corresponding DFT coefficients?



(A)



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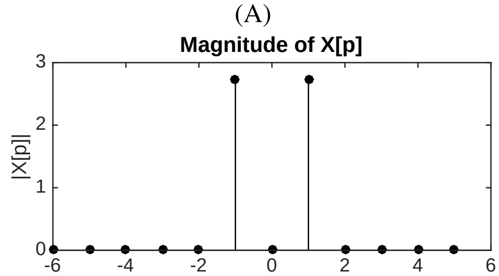

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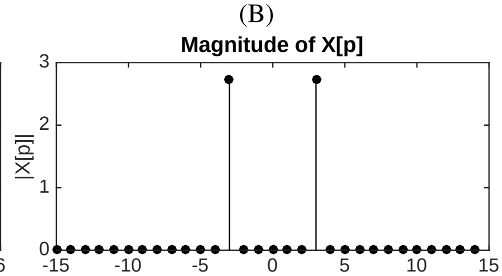
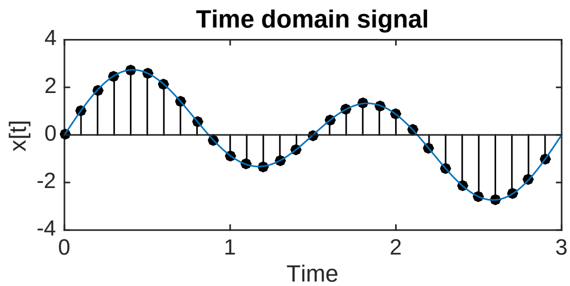
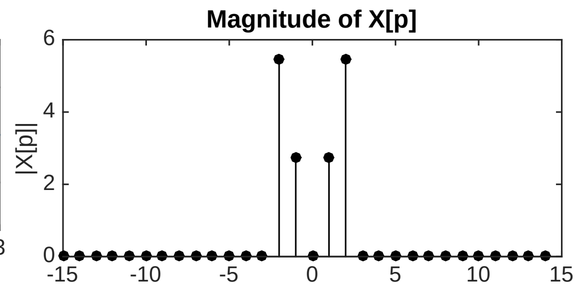


(D)

(b) Given the time domain signal below with 30 samples taken over 3 seconds,

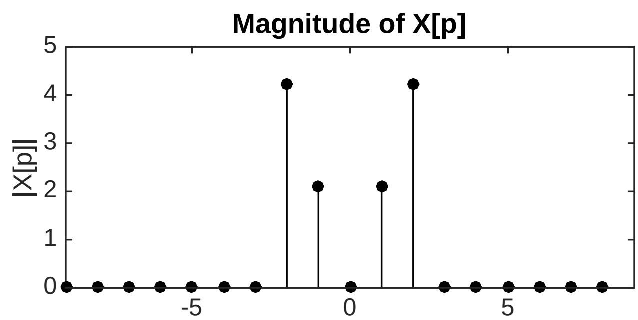


Time domain signal.

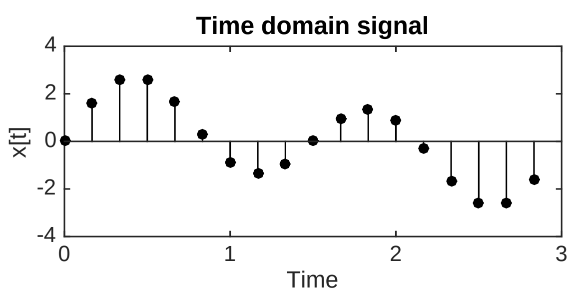


DFT magnitudes.

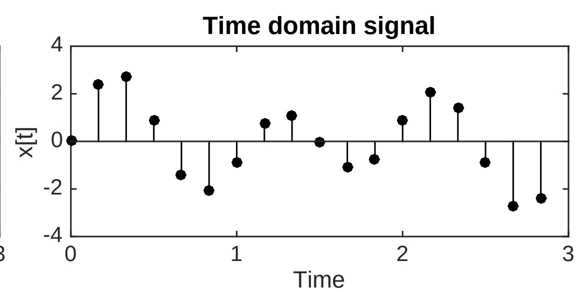
Now, we have obtained the DFT coefficients for another sampled signal, with sampling frequency of 6Hz:



which one is the corresponding sampled signal?



(A)



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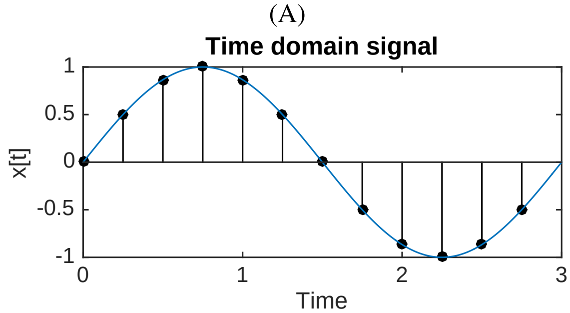

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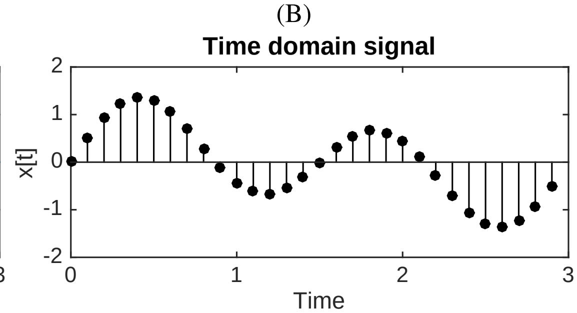
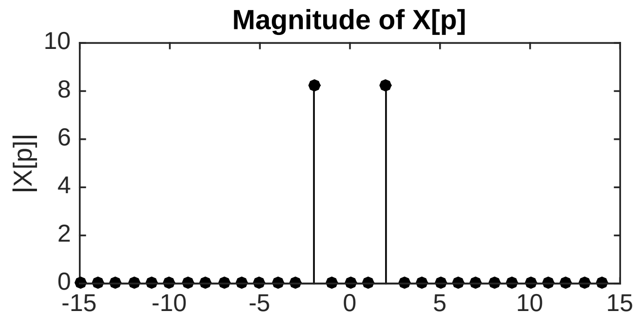
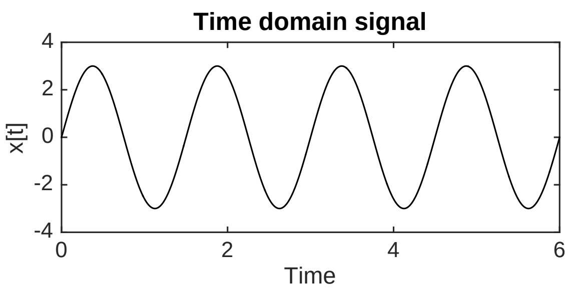


(D)

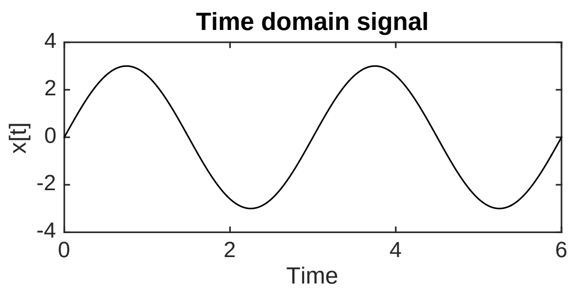
(c) Given the DFT coefficients of the sampled signal, obtained by taking 30 samples with a sampling frequency of 5Hz:



Which one is possibly the corresponding continuous time signal that was sampled?



(A)



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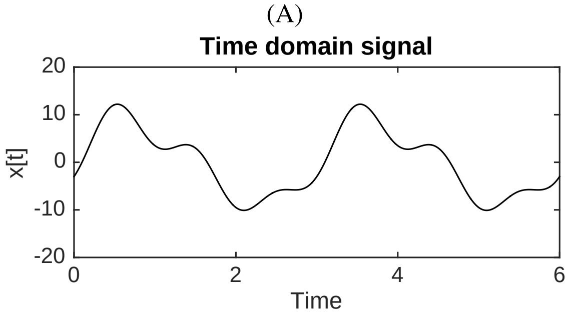
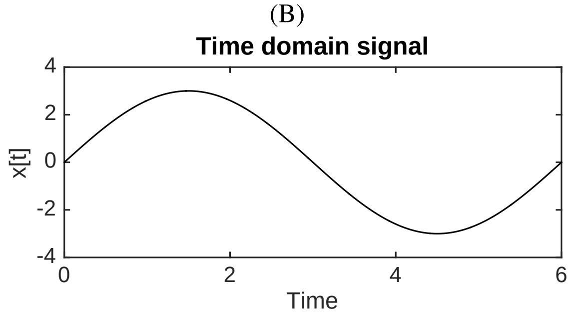


(C)



(D)

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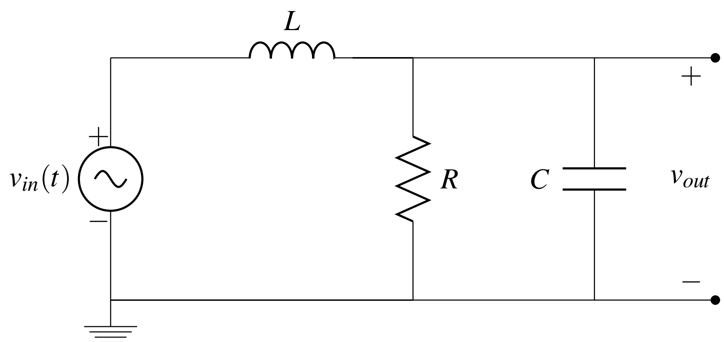
6. Analyze a circuit in the Phasor-domain (18pts)



The components in this circuit are described by:

$$v_{in}(t) = 10\cos(50t - 53^\circ)$$

$$R = 20\Omega, \quad L = 100mH, \quad C = 5mF$$

Here we assume $\cos(53^\circ) \approx \frac{3}{5}$ and $\sin(53^\circ) \approx \frac{4}{5}$ for the purposes of all calculations.

(a) (4 pts) **Express the voltage source as a phasor explicitly in the form** $a + bi$**.** Give numerical values for a and b.

(b) (8 pts) **Write down the transfer function** $H(\omega) = \frac{v_{out}}{v_{in}}$ **symbolically in terms of** C**,** R**,** L **and** ω**.**

(c) (6 pts) Since $v_{in}(t) = 10\cos(50t - 53^\circ)$, we know $v_{out}(t)$ can be written as $\alpha\cos(50t + \theta)$, where α is positive real number, and $0^\circ \le \theta < 360^\circ$. It turns out that $\alpha = 20\sqrt{2}$. **Compute the numerical value for θ.**

(HINT: Use the transfer function from the previous part appropriately.)

7. Frequency-Selective Filter (18 pts)

Recall that the $n = 4$ DFT basis can be expressed as columns of a matrix:

$$\begin{bmatrix} | & | & | & | \\ \vec{u}_0 & \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | & | \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

(a) (4 pts) Let $\vec{x}$ be a 4-length vector with each element of $\vec{x}$ defined by

$$\vec{x}[k] = 1 + \cos\frac{2\pi}{4}k$$

as $k = 0, 1, 2, 3$.

Find the DFT (frequency-domain representation) of $\vec{x}$.

(b) (8 pts) We want a real 4×4 matrix H that filters/projects any vector $\vec{y}$ onto the subspace spanned by $\vec{u}_0, \vec{u}_1, \vec{u}_3$. (i.e. $H\vec{y}$ returns a vector that is inside the subspace and is the closest point in the subspace to $\vec{y}$.) Furthermore, we want this H matrix to be circulant. (i.e. $H = C_{\vec{h}}$ for some $\vec{h}$). **What are the four eigenvalues and corresponding four eigenvectors for this matrix H?**

(c) (6 pts) **What is the first column of the circulant matrix H?**

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

8. SVD Me (9pts)

Compute the SVD (express a matrix as $A = U\Sigma V^T$ where U and V both have othonormal columns and Σ is diagonal matrix with non-negative entires) **of the matrix**

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

wherein V is the 3×3 identity matrix.

For your computational convenience:

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A^T A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

9. Observer/Estimator Design (14 pts)

Consider the linear discrete-time system whose vector state $\vec{x}$ evolves according to:

$$\vec{x}(t+1) = \begin{bmatrix} 0 & -2 \\ 3 & \frac{1}{3} \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

And has scalar output:

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{x}(t)$$

(a) (2 pts) Is this system observable?

(b) (12 pts) The observation $y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \vec{x}(t)$ gives us an observation of only one state variable. Let's build an observer/estimator to track both state variables. Define a system

$$\hat{x}(t+1) = \begin{bmatrix} 0 & -2 \\ 3 & \frac{1}{3} \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \vec{\ell}(y(t) - \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x}(t))$$

where $\vec{\ell} \in \mathbb{R}^2 = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$. Now let $\vec{e}(t) = \vec{x}(t) - \hat{x}(t)$ and the error dynamics become:

$$\vec{e}(t+1) = \underbrace{(\begin{bmatrix} 0 & -2 \\ 3 & \frac{1}{3} \end{bmatrix} - \vec{\ell} \begin{bmatrix} 0 & 1 \end{bmatrix})}_{\bar{A}} \vec{e}(t)$$

Find ℓ_1, ℓ_2 so that the error goes exactly to zero within two time steps no matter how bad our original estimate is. *HINT: $\bar{A}$ must have both eigenvalues be zero for this to be true.*

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 2: Free-form questions (68 + 24 pts)

You must show work for credit in all questions in this section.

10. Controllable Canonical Form (20 pts)

When we are trying to stabilize a robot, it is sometimes useful to put the dynamics into a standard form that lets us more easily adjust its behavior.

Consider the linear continuous-time system below.

$$\frac{d}{dt}\vec{s}(t) = A_r\vec{s}(t) + \vec{b}_r u(t) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \vec{s}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

(a) (10 pts) There exists a transformation $\vec{z}(t) = T\vec{s}(t)$ such that the resulting system is in controllable canonical form:

$$\frac{d}{dt}\vec{z}(t) = \begin{bmatrix} 0 & 1 \\ \alpha_0 & \alpha_1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Find this transformation matrix T and the resulting α_0 and α_1 in controllable canonical form.

HINT: The column vectors of T are just the standard basis vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ *arranged in some order.*

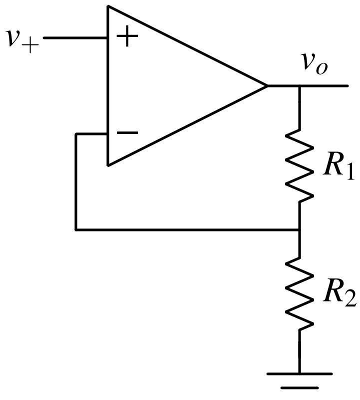
(b) (10 pts) For the system given by (A_r, $\vec{b}_r$, $\vec{s}$, u), **use the controllable canonical form from the previous part** to obtain a state feedback law $u(t) = \vec{f}^T\vec{z}(t) = \begin{bmatrix} f_0 & f_1 \end{bmatrix}\vec{z}(t)$ such that the resulting closed-loop system has eigenvalues $\lambda = -1, -2$ and then use the transformation T to get $u(t) = \vec{g}^T\vec{s}(t) = \begin{bmatrix} g_0 & g_1 \end{bmatrix}\vec{s}(t)$ a control law in terms of the original state variable $\vec{s}(t)$.

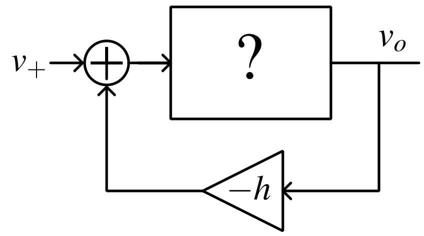
What are the vectors $\vec{f}$ and $\vec{g}$?

You will get full credit if you correctly use the properties of controllable canonical form to do this, but you may check your answer by another method if you so desire.

11. Feedback Control of Op-Amps (30 pts)

You have seen op-amps in negative feedback many times, and you have learned about feedback control. You may not have realized it yet, but these are actually related to each other.





Here, we introduce a dynamic model for a non-ideal op-amp:

$$\frac{d}{dt}v_o(t) = -v_o(t) + Gu(t)$$

where v_o is the output voltage, and $u(t) = (v_+(t) - v_-(t))$ where v_+ and v_- are the voltages at the positive and negative inputs respectively, and $G > 2$ is a parameter that defines the op-amp's behavior.

(a) (2 pts) Given the dynamic model for the nonideal op-amp, **assuming v_+ and v_- are not changing, for what value of v_o will v_o not be changing?** Your answer should depend on v_+, v_-, G.

(b) (2 pts) In the above, $v_-(t) = hv_o(t)$. **Pick values for the resistors R_1 and R_2 so that h equals $\frac{1}{2}$.**

(c) (8 pts) Suppose we place the nonideal op-amp in resistive negative feedback using a voltage divider whose ratio is $h = \frac{1}{2}$, in other words set

$$u(t) = v_+(t) - hv_o(t).$$

Write out the new differential equation that relates $v_o(t)$ to $v_+(t)$. Is this system stable? Briefly state why or why not.

(d) (4 pts) If we had swapped the roles of the positive and negative terminals of the op-amp (i.e. had hooked the nonideal op-amp up in positive feedback so that $u(t) = hv_o(t) - v_-(t)$.) **would the resulting closed-loop system have been stable? Briefly state why or why not.**

(e) (10 pts) For the closed-loop system in negative feedback (from part (c)) using resistor values that set $h = \frac{1}{2}$, assume that the output voltage starts at 0V at time 0 and that $v_+(t)$ was 0V but then jumps up to 1V at time 0. **How long will it take for $v_o(t)$ to reach 1V**? Your answer should be in terms of G.

(f) (4 pts) **What happens to the answer of the previous part if $G \to \infty$?**

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

12. Newton interpolation (18 pts)

We have studied polynomial interpolation using the monomial basis and the Lagrange basis. In this problem, we study a different basis, known as the Newton basis. This basis has the property that although it depends on the sampling points, each basis element does not depend on all of the sampling points. As you do this problem, you should begin to understand why this basis might be interesting if the samples are revealed to us one at a time in order.

Suppose we are given $n+1$ distinct points $x_0, x_1, \ldots, x_n$, and we sample a polynomial f at these points, such that we have the samples $f(x_0), f(x_1), \ldots, f(x_n)$. Our goal is to recover f given these samples.

We define the Newton polynomials as follows.

$$
\begin{aligned}
N_0(x) &= 1 \\
N_1(x) &= x - x_0 \\
&\vdots \\
N_n(x) &= (x - x_0)(x - x_1)\cdots(x - x_{n-1})
\end{aligned}
$$

Our goal is to find coefficients $a_0, \ldots, a_n$ such that

$$f(x) = a_0 N_0(x) + a_1 N_1(x) + \cdots + a_n N_n(x)\,.$$

(a) (8 pts) We know there must exist an $(n+1) \times (n+1)$ matrix U such that

$$U \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}.$$

Use what you know about the structure of U to compute a_0 and a_1 in terms of the x_i and $f(x_i)$. *(HINT: Write out the first two rows of U.)*

(b) (10 pts) Suppose we have $n = 2$, and our points are $x_0 = 0, x_1 = 1, x_2 = 2$, and $f(x_0) = 5$, $f(x_1) = -1$, $f(x_2) = 3$. **Find the coefficients** a_0, a_1, a_2**.**

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13. Efficient Cross Correlation (Bonus 24 pts)

Computing correlations is an important part of doing things like localization (as you saw in 16A) for GPS.

(a) (8 pts) Let $\vec{r}$ be the time-reversal of a real vector $\vec{x}$ so that $r[0] = x[0]$ and $r[k] = x[n-k]$ for $k = 1, \ldots, n-1$.
What is the relationship between the frequency-domain representation (DFT) of $\vec{r}$ — namely $\vec{R}$ — and the frequency-domain representation (DFT) of $\vec{x}$?

(b) (6 pts) The circular cross-correlation $\vec{z}$ of two n-dimensional real vectors $\vec{x}$ and $\vec{y}$ is defined by

$$z[t] = \sum_{k=0}^{n-1} x[k]y[k-t] \tag{1}$$

Where the indices arithmetic is taken mod-n. (So -3 is just $n-3$, etc.)
We can represent the cross correlation $\vec{z}$ as a matrix multiplication of a matrix $M_{\vec{x}}$ and the vector $\vec{y}$. **What is $M_{\vec{x}}$?**
(For purposes of exam grading, it would suffice to show us what $M_{\vec{x}}$ is for the case $n = 4$ but we actually have compact notation in the 16 series that can be used to express this matrix.)

(c) (10 pts) If you had a fast way of computing the DFT and Inverse-DFT, use that to **give a fast way of computing the circular cross-correlation $\vec{z}$ of $\vec{x}$ and $\vec{y}$?**

(HINT: Does $M_{\vec{x}}$ have any special structure that you can exploit? Can you compute with it fast in the frequency domain? Multiplication by a diagonal matrix is fast. Multiplication by a non-diagonal matrix is slow.)

[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.

Good luck on the rest of your final exams!]