MT3.1 (20 Points) This entire problem is restricted to the space of continuous-time signals that are periodic with fundamental period p and fundamental frequency $\omega_0 = 2\pi/p$. Each signal in this space may be complex-valued. Otherwise, the two parts of this problem are independent, so you may tackle them in either order.

(a) True or false? $||f+g||^2 + ||f-g||^2 = 2||f||^2 + 2||g||^2$.
Explain your reasoning.

Recall that $||f||^2 \triangleq \langle f, f\rangle \triangleq \int_{\langle p\rangle} f(t)\, f^*(t)\, dt$ and that the magnitude-squared of other such periodic functions is similarly defined.

$$||f+g||^2 + ||f-g||^2 = \langle f+g, f+g\rangle + \langle f-g, f-g\rangle$$
$$= \langle f, f\rangle + \langle g, f\rangle + \langle f, g\rangle + \langle g, g\rangle + \langle f, f\rangle - \langle g, f\rangle - \langle f, g\rangle + \langle g, g\rangle$$
$$= 2\langle f, f\rangle + 2\langle g, g\rangle = 2||f||^2 + 2||g||^2$$

true !!!

(b) Consider a signal x having the exponential Fourier series expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{ik\omega_0 t}$. Let

$$x_N(t) = \sum_{k=-N}^{+N} X_k\, e^{ik\omega_0 t}$$

be an approximation to x, and let $\varepsilon_N(t) = x(t) - x_N(t)$ denote an error signal. Prove that $\varepsilon_N \perp x_N$.

$$\varepsilon_N(t) = x(t) - x_N(t) = \sum_{k=-\infty}^{-N-1} X_k e^{ik\omega_0 t} + \sum_{k=N+1}^{\infty} X_k e^{ik\omega_0 t}$$

Since for continuous-time complex exponentials

$$\langle e^{ik\omega_0 t}, e^{il\omega_0 t}\rangle = p\,\delta(k-l),$$

where $\delta(n)$ is the Kronecker delta function, $k, l \in \mathbb{Z}$

$\langle \varepsilon_N, x_N\rangle = 0$ for there is no common frequency components

$\varepsilon_N \perp x_N$

MT3.2 (40 Points) Consider a periodic discrete-time signal x having fundamental period p and an exponential discrete Fourier series expansion

$$x(n) = \sum_{k=\langle p\rangle} X_k\, e^{ik\omega_0 n},$$

where ω_0 is the fundamental frequency of the signal. The signal has the following the properties:

- $x(n+3) = x(n),\ \forall n \in \mathbb{Z}$.
- $\sum_{n=\langle p\rangle} x(n) = 0$.
- $\sum_{k=\langle p\rangle} X_k = 0$.
- $\sum_{k=\langle p\rangle} |X_k|^2 = \frac{1}{2}$.

Show that the signal x can be expressed as $x(n) = A\cos(Bn + C)$, and determine the parameters A, B, and C.

Is there a unique answer? If so, explain. If not, determine at least two possible signals x and provide a well-labeled stem plot for each.

$x(n+3) = x(n)$, $p = 3$, $\omega_0 = \frac{2\pi}{3}$

$\sum_{n=\langle 3\rangle} x(n) = pX_0 = 3X_0 = 0$, $X_0 = 0$

Xk is imaginary

$\sum_{k=\langle 3\rangle} X_k = \sum X_k e^{ik\omega_0 n} = x(0) = 0 \Rightarrow x(-1) + x(1) = 0,\ x(-1) = -x(1)$

also $X_{-1} + X_1 = 0$, $X_1 = -X_{-1}$

From the Parseval's identity

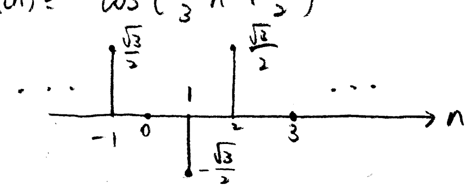
$\frac{1}{3}\sum_{n=\langle 3\rangle} |x(n)|^2 = \sum_{k=\langle 3\rangle} |X_k|^2 = \frac{1}{2}$, $2|x(1)|^2 = \frac{3}{2}$, $x(1) = \pm\frac{\sqrt{3}}{2}$ the answer is not unique

$2|X_1|^2 = \frac{1}{2}$ $|X_1| = \frac{1}{2}$

$x(n) = X_1 e^{i\frac{2\pi}{3}n} + X_{-1} e^{-i\frac{2\pi}{3}n} = Ae^{iC}e^{iBn} + Ae^{-iC}e^{-iBn} \Rightarrow A = 2|X_1| = 1$, $B = \frac{2\pi}{3}$, $C = \angle X_1 = \pm\frac{\pi}{2}$

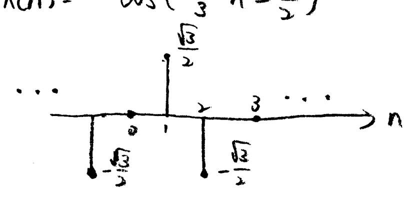
For $A = 1$, $B = \frac{2\pi}{3}$, $C = \frac{\pi}{2}$

$x(n) = \cos\left(\frac{2\pi}{3}n + \frac{\pi}{2}\right)$



For $A = 1$, $B = \frac{2\pi}{3}$, $C = -\frac{\pi}{2}$

$x(n) = \cos\left(\frac{2\pi}{3}n - \frac{\pi}{2}\right)$



MT3.3 (45 Points) Consider a discrete-time LTI filter H having impulse response h and frequency response H.

Recall that $H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}$.

A related discrete-time filter G has impulse response

$$g(n) = h(n) - h(n-1).$$

(a) Determine a simple expression for the frequency response values $G(\omega)$ in terms of $H(\omega)$.

$$G(\omega) = \sum_{n=-\infty}^{\infty} g(n)\, e^{-i\omega n} = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n} - \sum_{n=-\infty}^{\infty} h(n-1)\, e^{-i\omega n}$$

$$= H(\omega) - e^{-i\omega} \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}$$

$$= H(\omega) - e^{-i\omega} H(\omega) = H(\omega)(1 - e^{-i\omega})$$

(b) Suppose the frequency response of the filter H is given by

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \frac{1}{1 - 0.99\, e^{-i\omega}}.$$

(i) Determine a simple expression for its impulse response $h(n)$.

$$H(\omega) = \sum_{n=0}^{\infty} (0.99)^n\, e^{-i\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}$$

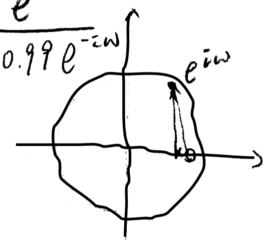
$$h(n) = (0.99)^n\, u(n)$$

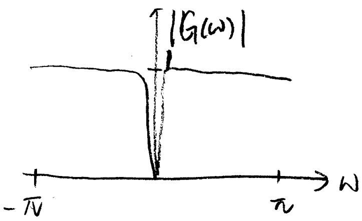
(b) **Continuation of part (b) from the previous page:**

(ii) **Determine a simple expression for the frequency response $G(\omega)$, and provide a well-labeled plot of the magnitude response $|G(\omega)|$. Specify whether the filter G is low-pass, band-pass, high-pass, notch, or none of these types. Explain your work.**

$$G(\omega) = 1 - e^{-i\omega} H(\omega) = \frac{1-e^{-i\omega}}{1-0.99e^{-i\omega}}$$

$$= \frac{e^{i\omega}-1}{e^{i\omega}-0.99}$$





it's a zero-frequency notch filter since the circle vector cancels the cross vector except at $\omega = 0$

(iii) **Determine the linear, constant-coefficient difference equation that governs the input x and output y of the filter G.**

$$G(\omega) = \frac{1-e^{-i\omega}}{1-0.99e^{-i\omega}}$$

$$(1-0.99e^{-i\omega})\, G(\omega) = 1 - e^{-i\omega}$$

$$y(n) - 0.99\,y(n-1) = x(n) - x(n-1) \Rightarrow y(n) = x(n) - x(n-1) + 0.99\,y(n-1)$$

(iv) **Determine a reasonably simple and accurate expression, as well as a well-labeled stem plot, for the response of the filter G to the input signal $x(n) = 1 + (-1)^n$? Explain.**

$$x(n) = 1 + e^{i\pi n}$$

$$y(n) = G(0) + G(\pi)\, e^{i\pi n}$$

$$= (-1)^n \quad \text{since } G(0) = 0,\ G(\pi) = 1$$