EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 22 September 2009
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Kompleckz FIRST Name Cardioid

Lab Time ?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided $8.5" \times 11"$ sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may or may not find the following information useful:

$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta. \\
\sin(\alpha+\beta) &= \sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta. \\
\cos\alpha+\cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right). \\
\sin\alpha+\sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right). \\
\cos\left(\frac{\pi}{3}\right) &= -\cos\left(\frac{2\pi}{3}\right)=\frac{1}{2}. \\
\sin\left(\frac{\pi}{3}\right) &= \sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}. \\
2^8 &= 256.
\end{aligned}
$$

MT1.1 (25 Points) In response to every input signal x, a discrete-time system produces a corresponding output signal y equal to the *two-point moving average* of the input. That is,

$$\forall n \in \mathbb{Z}, \quad y(n) = \frac{x(n) + x(n-1)}{2}.$$

We apply the following complex exponential input signal to the system:

$$\forall n \in \mathbb{Z}, \quad x(n) = e^{i\omega n}.$$

(a) Show that the output signal is specified by

$$\forall n \in \mathbb{Z}, \quad y(n) = H(\omega)\, e^{i\omega n}.$$

Express $H(\omega)$ in a reasonably simple form.

$$x(n) = e^{i\omega n} \implies x(n-1) = e^{i\omega(n-1)} = e^{-i\omega} e^{i\omega n} \implies$$

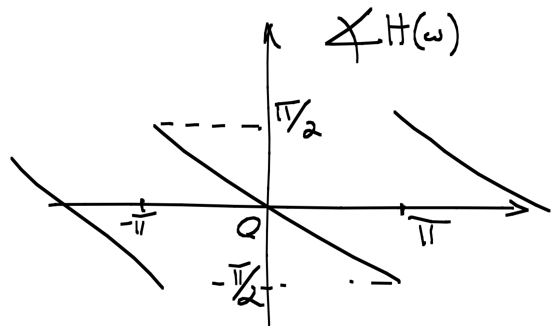
$$y(n) = \frac{e^{i\omega n} + e^{-i\omega} e^{i\omega n}}{2} = \frac{1 + e^{-i\omega}}{2} e^{i\omega n} = \underbrace{\frac{e^{i\omega/2} + e^{-i\omega/2}}{2} e^{-i\omega/2}}_{H(\omega)} e^{i\omega n}$$

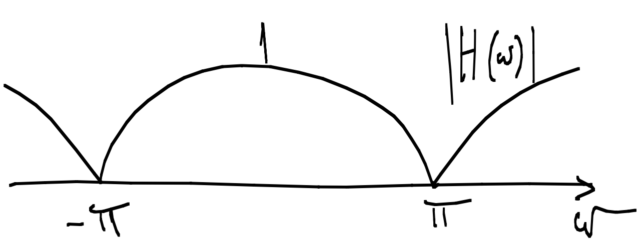
$$\implies H(\omega) = \cos\left(\frac{\omega}{2}\right) e^{-i\omega/2}$$

(b) Determine, and provide well-labeled plots of $|H(\omega)|$ and $\angle H(\omega)$, the magnitude and phase of $H(\omega)$, respectively.

$$|H(\omega)| = \left|\cos\left(\frac{\omega}{2}\right) e^{-i\omega/2}\right| = \left|\cos\left(\frac{\omega}{2}\right)\right| \underbrace{\left|e^{-i\omega/2}\right|}_{1} = \left|\cos\left(\frac{\omega}{2}\right)\right|$$

$$\measuredangle H(\omega) = \begin{cases} -\omega/2 & \text{if } \cos(\omega/2) > 0 \\ \pi - \omega/2 & \text{if } \cos(\omega/2) < 0 \end{cases}$$





Note: $H(\omega + 2\pi) = H(\omega)$

$$H(\omega + 2\pi) = \cos\left(\frac{\omega + 2\pi}{2}\right) e^{-i(\omega + 2\pi)/2} = H(\omega)$$

$$= \cos\left(\frac{\omega}{2} + \pi\right) e^{-i\pi} e^{-i\omega/2} = \cancel{-}\cos\left(\frac{\omega}{2}\right)(\cancel{-}1) e^{-i\omega/2}$$

MT1.2 (20 Points) The instantaneous position of a particle on the complex plane is described as follows:

$$\forall t \in \mathbb{R}, \quad x(t) = R(t)\, e^{i\theta(t)},$$

where

$$\theta(t) = \omega t,$$

for some constant, positive value of ω, and

$$R(t) = 1 + \cos\left[\theta(t)\right].$$

Provide a well-labeled plot of the trajectory of the particle on the complex plane. To receive credit, you must give a reasonable explanation as to how you obtained the plot.

Explain why it is reasonable to name the trajectory a *cardioid*.

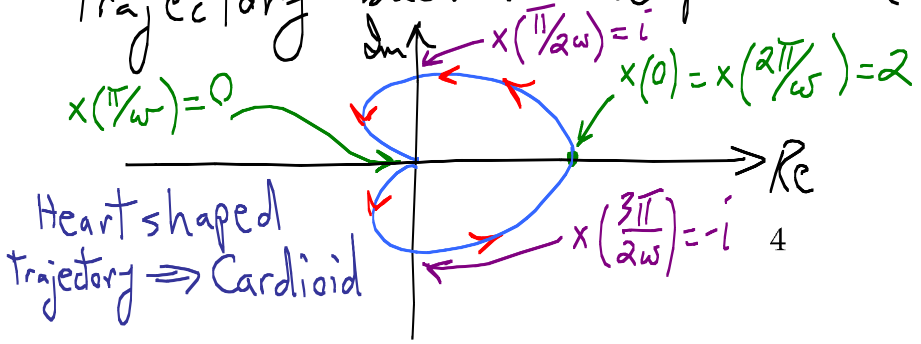
The modulus R is at its maximum if $\cos[\theta(t)]=1$, which requires $\theta(t)=2\pi k$, $k\in\mathbb{Z}$. In particular, $\theta(0)=0 \Rightarrow R(0)=2$. Similarly, if $\theta(T)=\omega T=2\pi$, that is, if $T=\frac{2\pi}{\omega} \Rightarrow R(0)=2$, and

$x(0)=x(T)=2$.

The modulus R is minimized if $\cos[\theta(t)]=-1$, which requires $\theta(t)=(2k+1)\pi$, $k\in\mathbb{Z}$. In particular, $\theta\left(\frac{\pi}{\omega}\right)=\omega\frac{\pi}{\omega}=\pi \Rightarrow$

$R(\pi/\omega)=0$.

As the particle attempts to rotate counterclockwise on the unit circle, its modulus decreases, until it reduces to zero @ $t=\pi/\omega$. As time progresses, the particle mirrors its trajectory back to the point $x(2\pi/\omega)=2$. If $\theta(t)=\frac{\pi}{2} \Rightarrow$ $R(t)=1$ and $x(t)=i$. This occurs @ $\omega t=2k\pi+\pi/2$, e.g., $t=\pi/2\omega=T/4$. Similarly, at $t=3T/4=3\pi/2\omega$, $x(t)=-i$.



Note: Particle spends equal time in each quadrant.

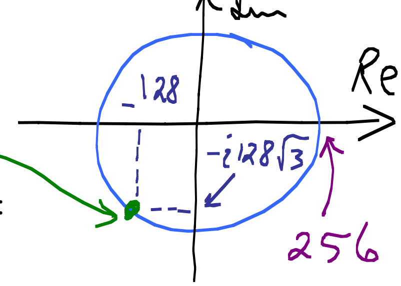
MT1.3 (15 Points) Evaluate $\left(-1+i\sqrt{3}\right)^8$ and identify it graphically on a well-label diagram of the complex plane.

Let $z = -1 + i\sqrt{3} = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2e^{i2\pi/3} \Rightarrow z^8 = 2^8 e^{i16\pi/3}$

But $e^{i16\pi/3} = e^{i\left(\frac{16\pi}{3} - \frac{12\pi}{3}\right)} = e^{i4\pi/3} \Rightarrow z^8 = \left(-1+i\sqrt{3}\right)^8 = 2^8 e^{i4\pi/3} = 256e^{i4\pi/3}$

That is: $\left(-1+i\sqrt{3}\right)^8 = 256e^{i4\pi/3} = 256\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -128 - i128\sqrt{3}$

$\left(-1+i\sqrt{3}\right)^8 = 256e^{i4\pi/3}$

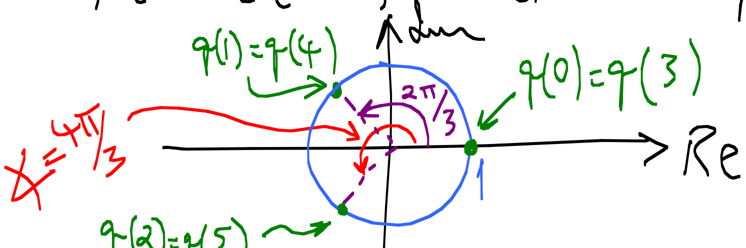


MT1.4 (25 Points) Consider two signals q and r described as follows:

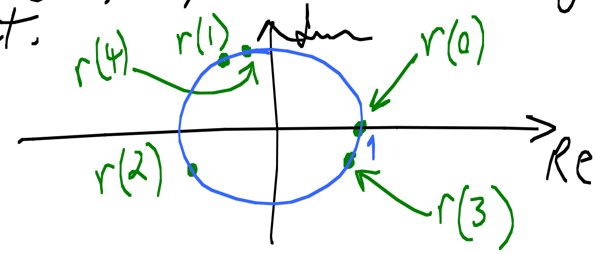
$$\forall n \in \mathbb{Z}, \qquad q(n) = e^{i2\pi n/3} \qquad r(n) = e^{i2n}.$$

(a) Provide a well-labeled, separate plot of each function for $n = 0, 1, 2, 3, 4$.

$q(0) = 1$, $q(1) = e^{i2\pi/3}$, $q(2) = e^{i4\pi/3}$, $q(3) = e^{i2\pi} = 1$, $q(4) = e^{i8\pi/3} = e^{i2\pi/3}$, ... retraces its points.



$r(0) = 1$, $r(1) = e^{i2}$, $r(2) = e^{i4}$, $r(3) = e^{i6}$, $r(4) = e^{i8}$, ..., doesn't retrace any point.



(b) For each function q and r determine how many *distinct* values the function may acquire for $n \in \mathbb{Z}$.

$q(n+3) = q(n) \Rightarrow$ Periodic with fundamental period 3 (three distinct points)

However, r is not a periodic function. That is, $\nexists\, p \in \{1, 2, 3, \ldots\}$ such that $r(n+p) = r(n)$. Hence, r has a countably infinite set of distinct values. Moreover, $r(n) \neq r(m)$ if $m \neq n$.

Sidenote: A DT signal $e^{i\omega_0 n}$ is periodic if, and only if, ω_0 is a rational multiple of π. Why?

MT1.5 (20 Points) Provide a well-labeled plot of the spectrum of the signal y described by $y(t) = x(t)\,c(t)$, for all t, where

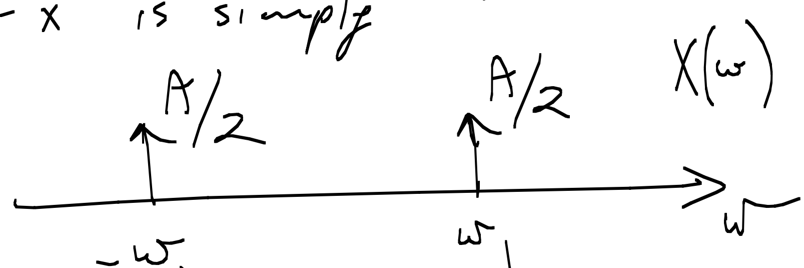
$$x(t) = \cos(\omega_1 t) \quad \text{and} \quad c(t) = \sum_{k=-\infty}^{+\infty} e^{ik\omega_0 t}.$$

Assume $0 < \omega_1 \ll \omega_0$.

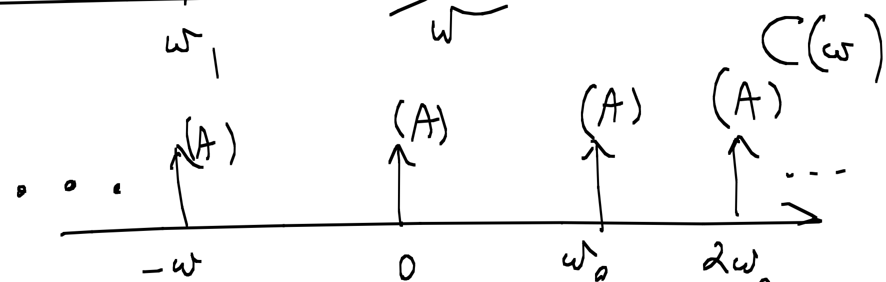
To receive credit, you must explain your reasoning.

The spectrum of a single-frequency exponential looks like the following: $e^{ik\omega_0 t}$

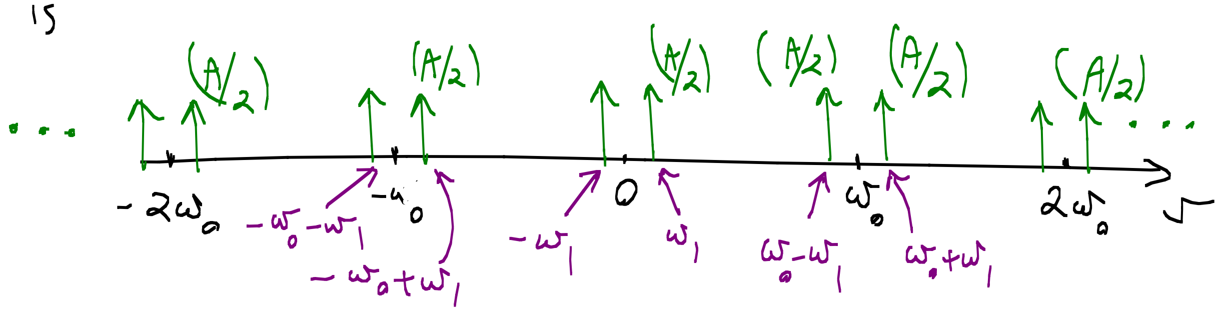
Noting that $x(t) = \cos(\omega_1 t) = \frac{e^{i\omega_1 t} + e^{-i\omega_1 t}}{2}$, the spectrum of x is simply



The spectrum of c is



So the spectrum of y is obtained by recognizing that $y(t) = x(t)c(t) = \frac{1}{2}\sum_k e^{i(\omega_1 + k\omega_0)t} + \frac{1}{2}\sum_k e^{i(-\omega_1 + k\omega_0)t}$. So the spectrum of y is



You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

LAST Name Kompleckz FIRST Name Cardioid

Lab Time ?

Problem	Points	Your Score
Name	10	10
1	25	25
2	20	20
3	15	15
4	25	25
5	20	20
Total	**115**	115