EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 3
Department of Electrical Engineering and Computer Sciences 25 April 2006
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Monster FIRST Name Frequency
Lab Time Before you know it!

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

Basic Formulas:

Discrete Fourier Series (DFS) Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, e^{-ik\omega_0 n} \, ,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\displaystyle\sum_{k=\langle p \rangle}$ may denote $\displaystyle\sum_{k=0}^{p-1}$ or $\displaystyle\sum_{k=1}^{p}$.

Continuous-Time Fourier Series (FS) Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{ik\omega_0 t} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \, e^{-ik\omega_0 t} dt \, ,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable continuous interval of length p. For example, $\displaystyle\int_{\langle p \rangle}$ can denote $\displaystyle\int_0^p$.

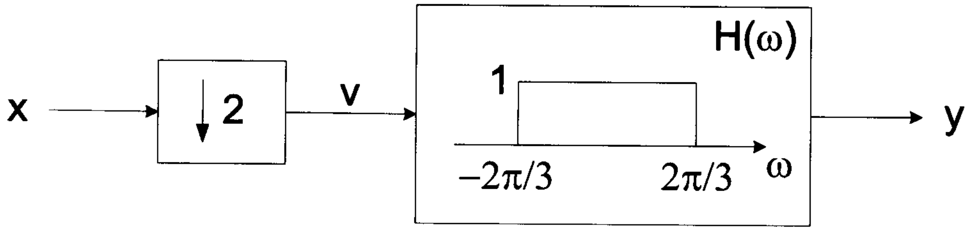
Frequency Response of a DT LTI System Consider a real, discrete-time LTI system having impulse response $h : \mathbb{Z} \to \mathbb{R}$. The frequency response $H : \mathbb{R} \to \mathbb{C}$ of the system is given by:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) \, e^{-i\omega n} \, , \forall \omega \, .$$

MT3.1 (20 Points) Consider a real, discrete-time signal x, which is periodic with period $p = 4$. The following is known about x:

$$x(n) = \begin{cases} 1/4 & n = 0 \\ 1/2 & n = 1 \\ 3/4 & n = 2 \\ 1 & n = 3\,. \end{cases}$$

As shown in the figure below, x is passed through a cascade of two systems: a down-sampler and an LTI discrete-time filter.



The output signal v of the downsampler is related to x as follows:

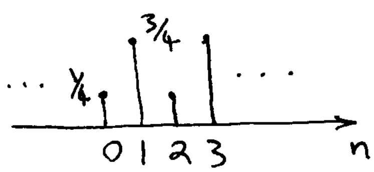
$$\forall n, \quad v(n) = x(2n).$$

The frequency response H of the filter is:

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \begin{cases} 1 & |\omega| \le 2\pi/3 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine the output signal y for all time samples n.

$v(n)$



v is periodic w/ period $p_v = 2$. Its fundamental frequency is $\omega_{0v} = \frac{2\pi}{p_v} = \frac{2\pi}{2} = \pi$

v has the following DFS expansion:

$$v(n) = \sum_{k=\langle 2\rangle} V_k e^{ik\omega_{0v}n} = V_0 + V_1 e^{i\pi n}.$$

The LTI filter eliminates the component of v at frequency π, and only passes the DC component. Hence, $y(n) = H(\omega)\big|_{\omega=0} V_0$

We know $V_k = \frac{1}{p_v} \sum_{n=\langle 2\rangle} v(n) e^{-ik\omega_{0v}n} \Rightarrow V_0 = \frac{1}{2}\sum_{n=\langle 2\rangle} v(n) = \frac{1}{2}\left(\frac{1}{4} + \frac{3}{4}\right) = \frac{1}{2} \Rightarrow y(n) = \frac{1}{2}$

MT3.2 (20 Points) Consider a continuous-time periodic signal $x : \mathbb{R} \to \mathbb{R}$ having period p, where $p > 0$. The signal x is passed through a cascade of two systems, as shown in the figure below:

$$\mathsf{x} \longrightarrow \boxed{\mathsf{F}} \xrightarrow{\mathsf{v}} \boxed{\mathsf{G}} \longrightarrow \mathsf{y}$$

The system $\mathsf{F} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$ is characterized by the following input-output relationship: $\forall t, v(t) = e^{-|x(t)|}$. The system $\mathsf{G} : [\mathbb{R} \to \mathbb{R}] \to [\mathbb{R} \to \mathbb{R}]$ is an LTI filter having a frequency response $G : \mathbb{R} \to \mathbb{C}$.

Select the strongest correct assertion from *each* of the categories shown below (i.e., you must select one assertion from each category). Justify each of your two selections succinctly, but clearly and convincingly.

Category I (a) The signal v is periodic with period at most p.

(b) The signal v is periodic with period $p_v = e^{-p}$.

(c) The signal v is periodic with period $p_v = |\ln p|$.

(d) The signal v is not periodic.

Category II (i) To determine $y(t), \forall t$, it is *sufficient* to know the frequency response values $G(\omega)$ ONLY at the discrete frequencies $0, \pm\frac{2\pi}{p_v}, \pm\frac{4\pi}{p_v}, \pm\frac{6\pi}{p_v}, \ldots$, where $p_v > 0$ is chosen appropriately.

(ii) To determine $y(t), \forall t$, it is *necessary* to know the frequency response values $G(\omega), \forall \omega$.

(iii) It is not possible to determine $y(t), \forall t$, based on the information given.

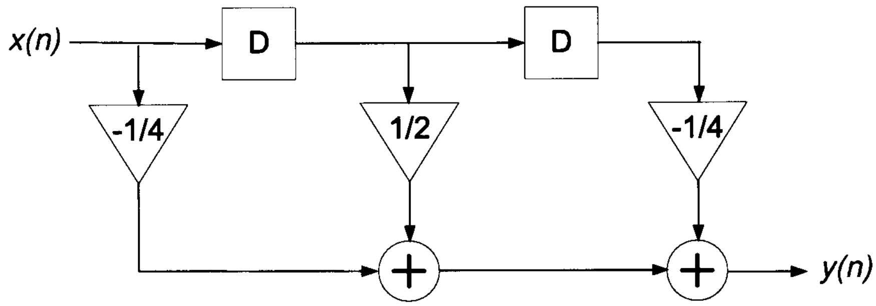
Answer: I(a) and II(a).

F is a memoryless system b/c $v(t)$ can be expressed as a function of $x(t)$: $v(t) = f(x(t))$, where $f(\cdot) = e^{-|\cdot|}$.
Hence, if x is periodic w/ period p, i.e., $x(t+p) = x(t)$, $\forall t$, the $v(t+p) = e^{-|x(t+p)|} = e^{-|x(t)|} = v(t)$, $\forall t$. It may be that p is not the smallest positive real number for which $v(t+p) = v(t)$. However, we know that p_v cannot exceed p. Since $v(t+p_v) = v(t)$, we can express v as follows: $v(t) = \sum_{k=-\infty}^{\infty} V_k e^{ik\omega_0 t}$, where $\omega_0 = \frac{2\pi}{p_v}$.
The output of the LTI filter G is $y(t) = \sum_{k=-\infty}^{\infty} V_k G(k\omega_0) e^{ik\omega_0 t}$.
Therefore, we need only know $G(\omega)\big|_{k\omega_0}$, $k \in \mathbb{Z}$.

MT3.3 (35 Points) Consider a causal, discrete-time LTI filter

$$\mathsf{F} : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$$

whose input-output relationship is characterized by the following delay-adder-gain block diagram:



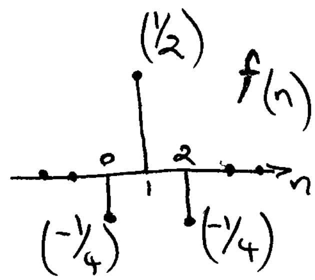
(a) Determine the linear, constant-coefficient difference equation governing the filter F.

$$y(n) = \frac{-1}{4} x(n) + \frac{1}{2} x(n-1) - \frac{1}{4} x(n-2)$$

(b) Determine the impulse response $f : \mathbb{Z} \to \mathbb{R}$ of the filter F.

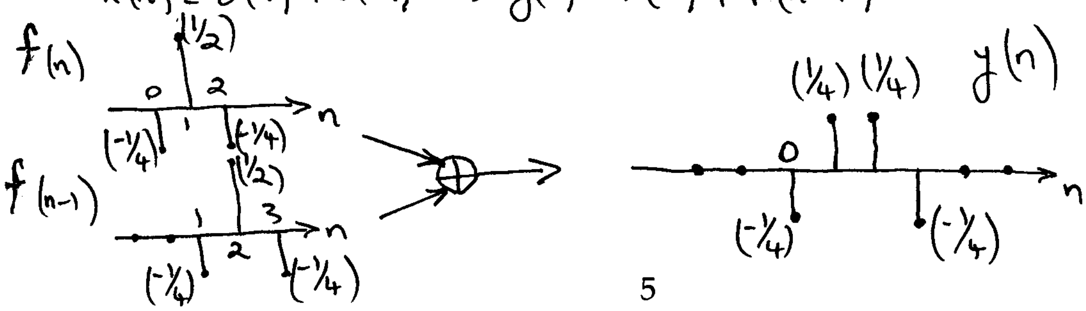
Let $x(n) = \delta(n) \implies$

$$f(n) = \frac{-1}{4}\delta(n) + \frac{1}{2}\delta(n-1) - \frac{1}{4}\delta(n-2)$$



(c) If $x(n) = \delta(n) + \delta(n-1), \forall n$, provide a well-labeled sketch of $y(n), \forall n$, the samples of the corresponding output signal y.

$x(n) = \delta(n) + \delta(n-1) \implies y(n) = 1 \cdot (n) + 1 \cdot (n-1)$

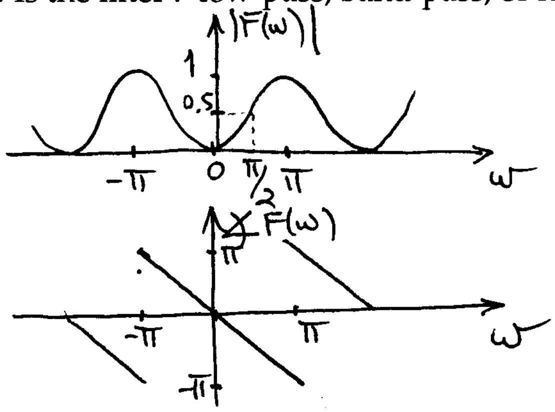


(d) Determine a simple expression for the frequency response $F : \mathbb{R} \to \mathbb{C}$ of the filter F, i.e., express $F(\omega)$ as a function of the frequency variable ω.

$$F(\omega) = -\frac{1}{4} + \frac{1}{2}e^{-i\omega} - \frac{1}{4}e^{-i2\omega} = -\frac{1}{4}e^{-i\omega}\left(e^{i\omega} + e^{-i\omega}\right) + \frac{1}{2}e^{-i\omega}$$

$$= -\frac{1}{2}\cos\omega\, e^{-i\omega} + \frac{1}{2}e^{-i\omega} = \underbrace{\frac{1}{2}\left[1 - \cos\omega\right]}_{|F(\omega)|}\underbrace{e^{-i\omega}}_{e^{i\measuredangle F(\omega)}}$$

(e) For the frequency range $-\pi < \omega \leq +\pi$, provide well-labeled sketches of $|F(\omega)|$ and $\angle F(\omega)$, the magnitude response and phase response of the filter F, respectively. Is the filter F low-pass, band-pass, or high-pass?



F is a high-pass filter

(f) Suppose the input signal x is characterized by

$$\forall n, \quad x(n) = \cos\left(\frac{7\pi}{3}n + \frac{\pi}{3}\right).$$

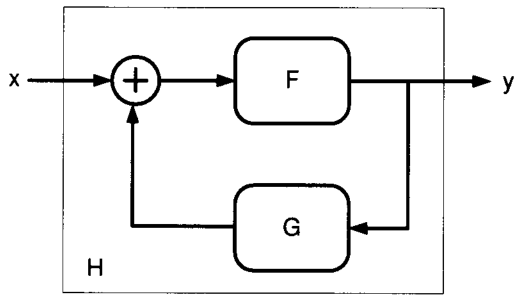
Determine a simple expression for $y(n), \forall n$, the sample values of the corresponding output signal y.

$$x(n) = \cos\left(\frac{7\pi}{3}n + \frac{\pi}{3}\right) = \cos\left(\frac{7\pi}{3}n - 2\pi n + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}n + \pi\right)$$

$$y(n) = \left|F\left(\frac{\pi}{3}\right)\right| \cos\left(\frac{\pi}{3}n + \frac{\pi}{3} + \measuredangle F\left(\frac{\pi}{3}\right)\right)$$

$$\left.\begin{array}{l} \left|F\left(\frac{\pi}{3}\right)\right| = \frac{1}{2}\left[1 - \cos\frac{\pi}{3}\right] = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4} \\ \measuredangle F\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} \end{array}\right\} \Rightarrow y(n) = \frac{1}{4}\cos\left(\frac{\pi}{3}n\right)$$

MT3.4 (30 Points) A feedback composition H of real, discrete-time LTI systems F and G is shown below:



The frequency response $F : \mathbb{R} \to \mathbb{C}$ of the system F is

$$\forall \omega \in \mathbb{R}, \quad F(\omega) = \frac{\frac{1}{4}e^{i\omega} + \frac{1}{2} + \frac{1}{4}e^{-i\omega}}{1+2\,e^{i\omega N}},$$

where $N \in \mathbb{N}$ and $N \geq 3$. The frequency response $G : \mathbb{R} \to \mathbb{C}$ of the system G is

$$\forall \omega \in \mathbb{R}, \quad G(\omega) = \frac{1}{\frac{1}{4}e^{i\omega} + \frac{1}{2} + \frac{1}{4}e^{-i\omega}}.$$

(a) Determine a simple expression for $H(\omega)$ characterizing the frequency response $H : \mathbb{R} \to \mathbb{C}$ of the composite feedback system H.

See the last page of this PDF file for the solution to this part.

(b) Determine the impulse response h of the composite feedback system H.

$h(n) = \frac{1}{2}\,\delta(n-N)$

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Problem	Points	Your Score
Name	10	10
1	20	20
2	20	20
3	35	35
4	30	30
Total	**115**	115

Correction for 3.4(a)

MT 3.4

(a)

$$
\begin{aligned}
H(\omega) &= \frac{F(\omega)}{1 - F(\omega)G(\omega)} \\
F(\omega)G(\omega) &= \frac{1}{1 + 2e^{i\omega N}} \\
1 - F(\omega)G(\omega) &= 1 - \frac{1}{1 + 2e^{i\omega N}} \\
&= \frac{1 + 2e^{i\omega N} - 1}{1 + 2e^{i\omega N}} \\
&= \frac{2e^{i\omega N}}{1 + 2e^{i\omega N}} \\
H(\omega) &= \frac{1 + 2e^{i\omega N}}{2e^{i\omega N}} \times \frac{\frac{1}{4}e^{i\omega} + \frac{1}{2} + \frac{1}{4}e^{-i\omega}}{1 + 2e^{i\omega N}} \\
&= \frac{\frac{1}{4}e^{i\omega} + \frac{1}{2} + \frac{1}{4}e^{-i\omega}}{2e^{i\omega N}} \\
&= \frac{1}{8}e^{-i\omega(N-1)} + \frac{1}{4}e^{-i\omega N} + \frac{1}{8}e^{-i\omega(N+1)}
\end{aligned}
$$