**EECS 20N: Structure and Interpretation of Signals and Systems** FINAL S07
Department of Electrical Engineering and Computer Sciences 16 May 2007
UNIVERSITY OF CALIFORNIA BERKELEY

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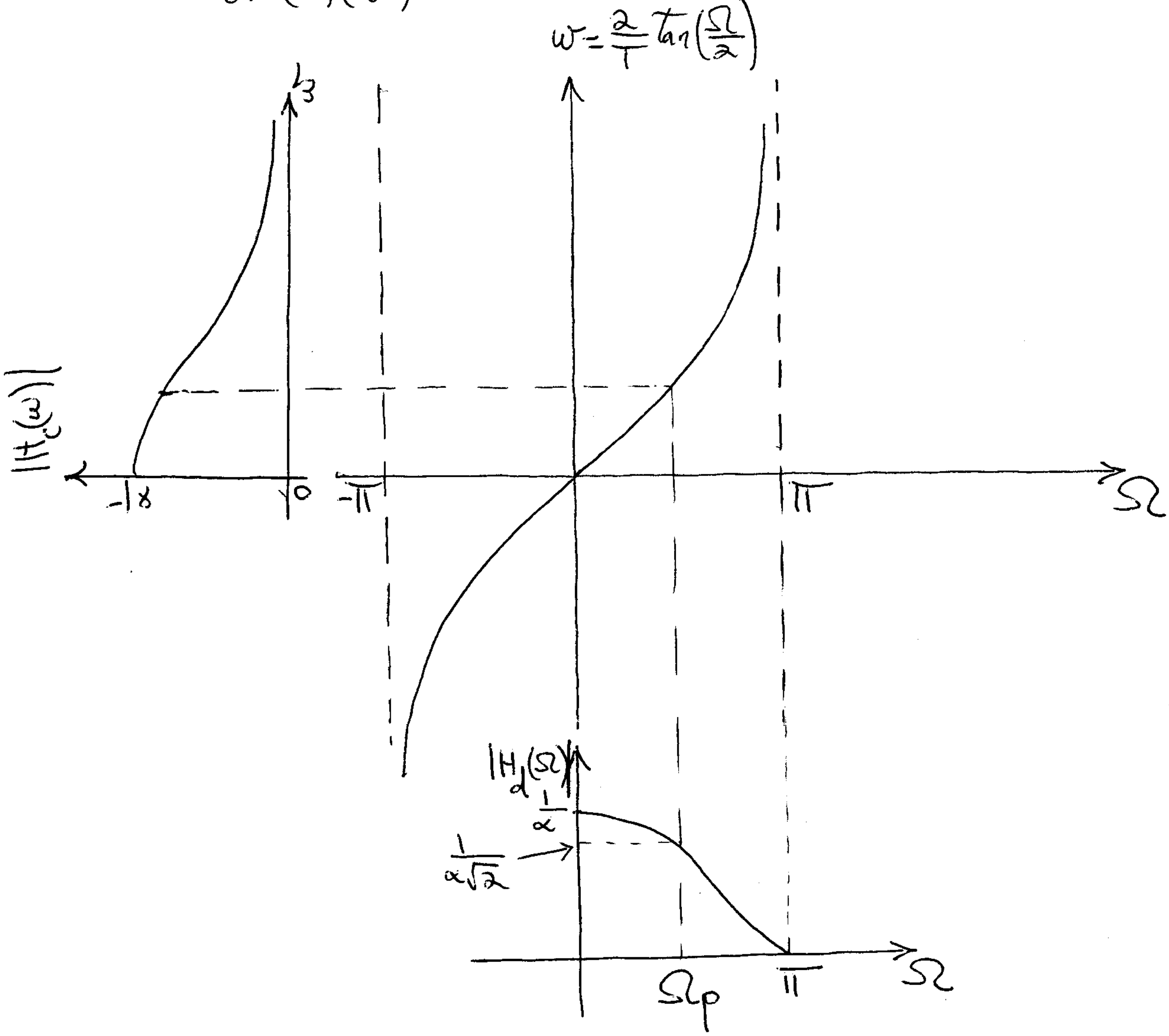
Lab Time Any time

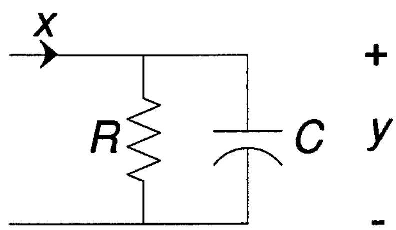
- **(10 Points)** Print your name and lab time in legible, block lettering above (5 points) AND on the last page (5 points) where the grading table appears.
- This exam should take up to 120 minutes to complete. You will be given at least 120 minutes, up to a maximum of 170 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except four double-sided 8.5" $\times$ 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 12.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the twelve numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

F-S07.2 (b)(ii)



**F-S07.1 (60 Points)** In this problem, you will explore the time- and frequency-domain descriptions of the input-output behavior of the first-order $RC$-circuit shown below




The signal $x$ represents the input current. The signal $y$ is the output of the system, and it denotes the voltage drop across the resistor-capacitor parallel combination.

The resistance $R$ and the capacitance $C$ are real constants having units of Ohms and Farads, respectively.

Throughout this problem, assume that the circuit is at initial rest; that is, the initial charge on the capacitor is zero.

The circuit can be considered a continuous-time LTI system having impulse response $h : \mathbb{R} \to \mathbb{R}$ and frequency response $H : \mathbb{R} \to \mathbb{C}$.

You have discovered that if the input current is

$$x(t) = e^{i\omega t}, \quad \forall t,$$

for any value of the frequency variable $\omega \in \mathbb{R}$, then the output voltage is

$$y(t) = \frac{R}{i\omega RC + 1} e^{i\omega t}, \quad \forall t.$$

(a) Determine a simple expression for $h(t)$, $\forall t \in \mathbb{R}$, the value of the impulse response $h$ at time $t$.

By the eigenfunction property of complex exponentials, we recognize that $H(\omega) = \dfrac{R}{i\omega RC+1} \Rightarrow H(\omega) = \dfrac{1}{C}\dfrac{1}{i\omega + \frac{1}{RC}} \Longrightarrow$

$h(t) = \dfrac{1}{C} e^{-t/RC} u(t)$ | Recall: $e^{-\alpha t} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{1}{i\omega + \alpha}$ $\quad Re(\alpha) > 0$

(b) Determine the linear, constant-coefficient differential equation governing the input $x$ and the output $y$.

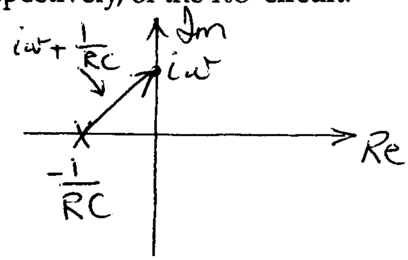
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{R}{i\omega RC+1} \Longrightarrow (i\omega RC+1)Y(\omega) = R\,X(\omega) \Longrightarrow$$

Using the differentiation property of the Fourier transform we arrive at the differential equation:

$$RC\,\dot{y}(t) + y(t) = R\,x(t)$$

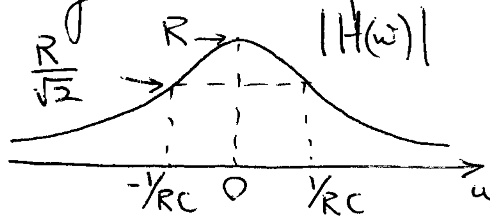
(c) Provide well-labeled plots of $|H(\omega)|$ and $\angle H(\omega)$, the magnitude and phase response values, respectively, of the $RC$-circuit.

$$H(\omega) = \frac{1/C}{i\omega + \frac{1}{RC}}$$



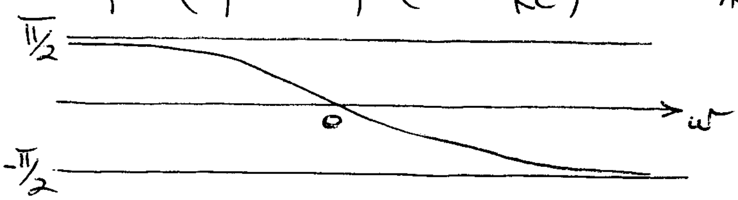

$H(0) = R$

Using the graphical method, we find the magnitude response:




$$\measuredangle H(\omega) = \cancel{\measuredangle \tfrac{1}{C}}^{\,0} - \measuredangle\left(i\omega + \tfrac{1}{RC}\right) \Longrightarrow$$

$$\measuredangle H(\omega) = -\measuredangle\left(i\omega + \tfrac{1}{RC}\right) = -\tan^{-1}\frac{\omega}{1/RC} = -\tan^{-1}\omega RC$$




(d) If the circuit is considered an analog filter, what type of filter would it be? Low-pass, band-pass, high-pass, notch, anti-notch, or none of these? Explain your reasoning succinctly, but clearly and convincingly.

Clearly a low-pass filter: favors lower frequencies near zero over higher frequencies away from $\omega = 0$.

**F-S07.2 (60 Points)** You have been commissioned to design a discrete-time LTI filter. The main requirement is that your filter must be based on an already-designed continuous-time LTI filter having impulse response and frequency response values

$$h_c(t) = e^{-\alpha t} u(t),\ \forall t \in \mathbb{R} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H_c(\omega) = \frac{1}{i\omega + \alpha},\ \forall \omega \in \mathbb{R},$$

where $\alpha > 0$.

Let $h_d$ denote the impulse response, and $H_d$ the frequency response, of the discrete-time filter that you have to design.

(a) You pore over filter-design literature and ... voilà! ... you feel like you have struck gold when you come across *impulse invariance,* a method of designing a discrete-time filter from a continuous-time filter. "Exactly the kind of information I was looking for!" you exclaim.

You discover that impulse invariance starts with an already-designed analog filter whose impulse response values are $h_c(t)$, and whose frequency response values are $H_c(\omega)$.

The discrete-time filter's impulse response values $h_d(n)$ are then defined as

$$h_d(n) = T\, h_c(nT), \quad \forall n \in \mathbb{Z},$$

where $T > 0$ is a "sampling" period.

Do NOT worry! You need NOT have any knowledge of sampling theory to work through this project!

(i) Determine a simple expression for the discrete-time filter's impulse response values $h_d(n)$.

$$\left.\begin{aligned} h_c(t) &= e^{-\alpha t} u(t) \\ u_c(t) &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \end{aligned}\right\} \Rightarrow h_d(n) = T h_c(nT) = T e^{-\alpha T n} u_c(nT) \Rightarrow$$

$u_c(nT) = u_d(n)$ b/c $T > 0$ and $u(0) = 1$.

$$\underbrace{h_d(n) = T e^{-\alpha T n} u_d(n)}\ ,\ \text{where}$$

$$u_d(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$ is the discrete-time unit-step function.

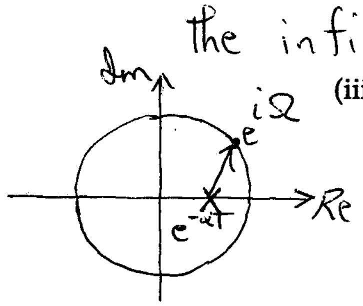
(ii) Determine a simple expression for the discrete-time filter's frequency response values $H_d(\Omega)$.
Also provide a well-labeled *magnitude* response plot of your discrete-time filter.
What type of discrete-time filter have you designed? Low-pass, band-pass, high-pass, notch, anti-notch, or none of these? Explain your reasoning succinctly, but clearly and convincingly.

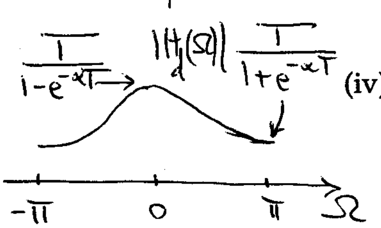
$$H_d(\Omega) = \sum_{n=-\infty}^{\infty} h_d(n) e^{-i\Omega n}$$

$$h_d(n) = T\left(e^{-\alpha T}\right)^n u(n)$$

$$\Longrightarrow H_d(\Omega) = T\sum_{n=0}^{\infty}\left(e^{-\alpha T} e^{-i\Omega}\right)^n = \frac{T}{1 - e^{-\alpha T} e^{-i\Omega}}$$

Note that $0 < e^{-\alpha T} < 1$ because $\alpha > 0, T > 0 \Longrightarrow$ the infinite sum converges. For plot, use the graphical techique.




Low-pass filter

(iii) Is your discrete-time filter causal? Explain your reasoning succinctly, but clearly and convincingly.

Yes, the filter is causal because $h_d(n) = 0, \; \forall n < 0$.

(iv) Suppose the input signal applied to your discrete-time filter is $x_d(n) = (-1)^n, \forall n$. Determine a simple expression for the corresponding output values $y_d(n)$.

$$x_d(n) = (-1)^n = e^{i\pi n} \Longrightarrow y_d(n) = H_d(\pi) e^{i\pi n}$$

$$\Longrightarrow y_d(n) = \frac{T}{1+e^{-\alpha T}} (-1)^n$$

(b) In a conversation with one of your colleagues, you learn about another method of designing a discrete-time filter from a continuous-time filter.

The method employs a mathematical mapping called the *bilinear transformation,* and it begins with an already-designed analog filter having frequency response $H_c$. The discrete-time filter is then obtained by letting

$$i\omega = \frac{2}{T}\frac{1-e^{-i\Omega}}{1+e^{-i\Omega}} \tag{1}$$

in the frequency response expression $H_c(\omega)$. Here, $T > 0$ is a parameter whose value is chosen appropriately (e.g., based on convenience).

Note that the "continuous-time" frequency $\omega$ has units of radians per second, whereas the "discrete-time" frequency $\Omega$ has units of radians per sample. In this context, both frequencies enter the equations, so we distinguish them by using different symbols for them.

The frequency response $H_d$ of the discrete-time filter is obtained from its continuous-time counterpart according to the equation

$$H_d(\Omega) = H_c(\omega)\big|_{\omega=\frac{2}{iT}\frac{1-e^{-i\Omega}}{1+e^{-i\Omega}}}\,.$$

(i) Use the bilinear transformation of Equation (1) to establish the following relationship between the two frequency variables:

$$\omega = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right) \quad \text{or, equivalently,} \quad \Omega = 2\arctan\left(\frac{\omega T}{2}\right).$$

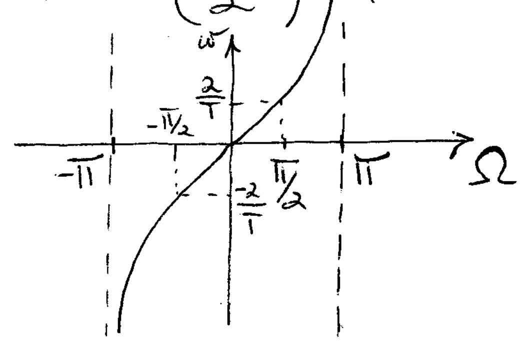
Provide a well-labeled plot of $\omega$ as a function of $\Omega$.

$$\cancel{i}\omega = \frac{2}{T}\frac{1-e^{-i\Omega}}{1+e^{-i\Omega}} = \frac{2}{T}\frac{\cancel{e^{-i\Omega/2}}\left(e^{i\Omega/2}-e^{-i\Omega/2}\right)}{\cancel{e^{-i\Omega/2}}\left(e^{i\Omega/2}+e^{-i\Omega/2}\right)} = \frac{2}{T}\frac{\cancel{2i}\sin(\Omega/2)}{\cancel{2}\cos(\Omega/2)}$$

$$\Rightarrow \omega = \frac{2}{T}\frac{\sin(\Omega/2)}{\cos(\Omega/2)} = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right) \Rightarrow \omega = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right)$$

$$\Rightarrow \frac{\omega T}{2} = \tan\left(\frac{\Omega}{2}\right) \Rightarrow \frac{\Omega}{2} = \arctan\left(\frac{\omega T}{2}\right) \Rightarrow$$

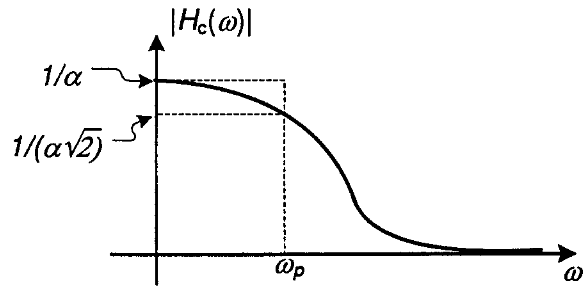
$$\Omega = 2\arctan\left(\frac{\omega T}{2}\right)$$



(ii) Recall the continuous-time filter described at the start of this problem:

$$h_c(t) = e^{-\alpha t}u(t),\ \forall t \in \mathbb{R} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H_c(\omega) = \frac{1}{i\omega + \alpha},\ \forall \omega \in \mathbb{R},$$

where $\alpha > 0$. The magnitude response $|H_c(\omega)|$ of this filter is shown below.




The frequency $\omega_p$, corresponding to the minimum pass-band gain $1/(\alpha\sqrt{2})$, defines the filter's pass-band cutoff frequency.

Determine $\omega_p$. $\quad H_c(\omega_p) = \frac{1}{\alpha\sqrt{2}}$ ; this happens

for $\underline{\omega_p = \alpha}$ $\quad H_c(\alpha) = \frac{1}{\alpha + i\alpha} \Longrightarrow |H_c(\alpha)| = \frac{1}{\sqrt{\alpha^2+\alpha^2}} = \frac{1}{\alpha\sqrt{2}}$

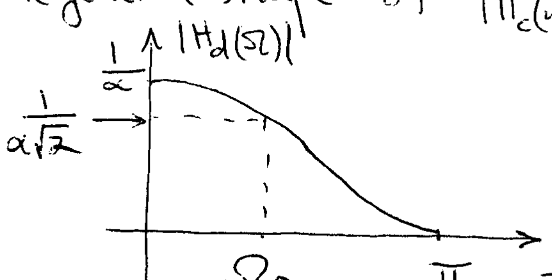
Provide a well-labeled plot of the magnitude response $|H_d(\Omega)|$ of the discrete-time filter obtained by applying the bilinear transformation of Equation 1 to the continuous-time filter above. Be sure to show an expression for $\Omega_p$, the discrete-time frequency to which $\omega_p$ maps.

Notice that the bilinear transformation does not cause an amplitude/magnitude scaling

$$H_d(\Omega) = H_c(\omega)\Big|_{\omega = \frac{2}{iT}\frac{1-e^{-i\Omega}}{1+e^{-i\Omega}}}$$

Rather, it causes only a frequency warping: $\Omega = 2\arctan\left(\frac{\omega T}{2}\right)$

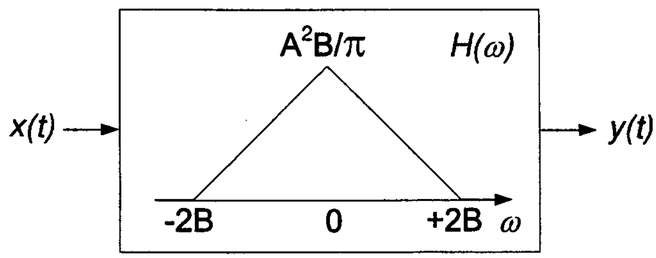
So the general shape of $|H_c(\omega)|$ is preserved → Magnitudes Preserved
→ Frequencies warped




To see the details of this frequency warping, see p. 2.

$\Omega_p = 2\arctan\left(\frac{\omega_p T}{2}\right) = 2\arctan\left(\frac{\alpha T}{2}\right)$

**F-S07.3 (50 Points)** The figure below shows a continuous-time LTI filter having a triangular frequency response $H$ as indicated. The impulse response of the filter is $h$, and the parameters $A$ and $B$ are positive constants.




(a) True or false? The impulse response $h$ is a real-valued and even function of time $t$. Explain your reasoning succinctly, but clearly and convincingly.

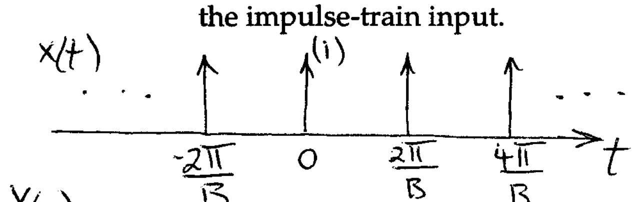
True. $H(\omega) \in \mathbb{R}$ $\quad H(\omega) = H(-\omega) \implies h(t) \in \mathbb{R} \wedge h(t) = h(-t)$

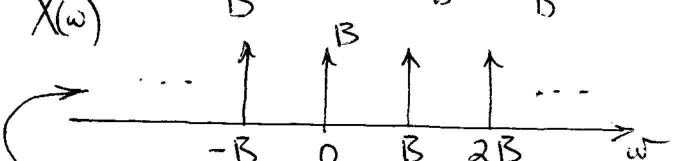
In words: The Fourier transform of $h$ is real and even if, and only if, $h$ is real and even.

(b) Suppose the input signal applied to this filter is the infinite-duration impulse train characterized by

$$\forall t \in \mathbb{R}, \quad x(t) = \sum_{n=-\infty}^{+\infty} \delta\left(t - \frac{2\pi n}{B}\right).$$

Determine a simple expression for $y(t)$, the output of the filter in response to the impulse-train input.




Derivation on the right.

$x$ is periodic with period $p = \frac{2\pi}{B}$

$\implies x$ has a Fourier series expansion $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t}$,

where $\omega_0 = \frac{2\pi}{p} = B$ is the fundamental frequency.

$\Downarrow$

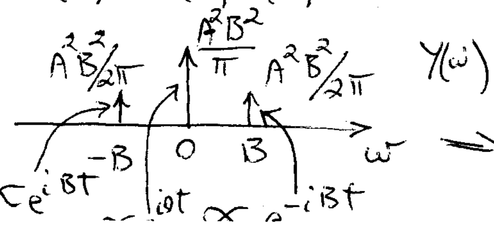
$$X_k = \frac{1}{p} \int_{-p/2}^{p/2} x(t) e^{-ik\omega_0 t} dt = \frac{B}{2\pi} \int_{-\pi/B}^{\pi/B} \delta(t) e^{-ik\omega_0 t} dt$$

$$X_k = \frac{B}{2\pi} \implies$$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} X_k \delta(\omega - k\omega_0) = B \sum_{k=-\infty}^{\infty} \delta(\omega - kB)$$

$y(t) = (x * h)(t) \implies$

$Y(\omega) = X(\omega) H(\omega) \implies$



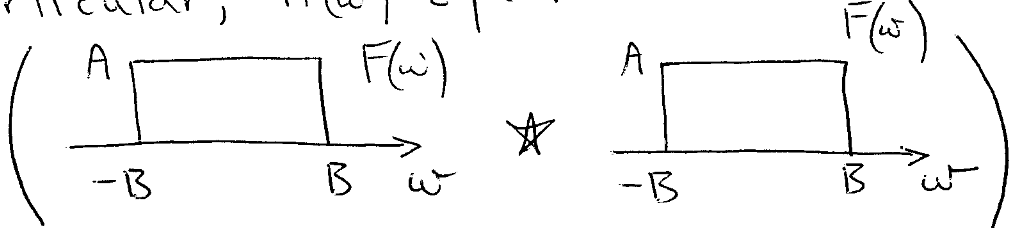

$\propto e^{iBt}$ $\quad \propto e^{i0t}$ $\quad \propto e^{-iBt}$

$$\implies y(t) = \frac{A^2B^2}{2\pi^2} + \frac{A^2B^2}{2\pi^2}\cos(Bt) \implies y(t) = \frac{A^2B^2}{2\pi^2}(1 + \cos Bt)$$

(c) Determine an expression for $h(t)$, the impulse response of the filter. Choosing the appropriate properties of the Fourier transform and convolution can drastically reduce your work here.

The triangular $H(\omega)$ is obtainable by the convolution of two "box functions" in the frequency domain. In particular, $H(\omega)$ equals

$\frac{1}{2\pi} \times$




scalar multiplication.

That is, $H(\omega) = \frac{1}{2\pi}(F \star F)(\omega)$. Therefore, in the time domain,

$$h(t) = f(t) f(t) = f^2(t).$$

Obtaining $f(t)$ is easy (we know it must be a sinc-like function).

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega = \frac{A}{2\pi}\int_{-B}^{B} e^{i\omega t}\, d\omega = \frac{A}{2\pi}\,\frac{e^{iBt} - e^{-iBt}}{it} \Rightarrow$$

$$f(t) = \frac{A}{\pi t}\sin(Bt) = \frac{AB}{\pi}\operatorname{sinc}\left(\frac{Bt}{\pi}\right) \Rightarrow$$

$$h(t) = \frac{A^2}{\pi^2 t^2}\sin^2(Bt)$$

**F-S07.4 (20 Points)** The autocorrelation function of a signal $f : \mathbb{R} \to \mathbb{R}$ is defined as

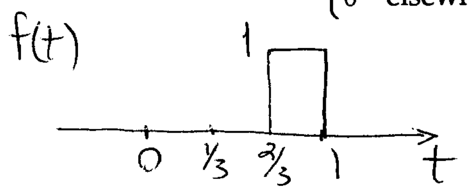
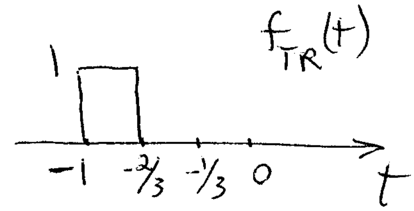
$$\phi_{ff}(t) = \int_{-\infty}^{+\infty} f(\tau)\, f(\tau - t)\, d\tau,$$

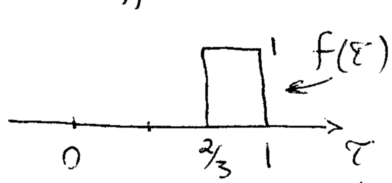
(a) Determine an expression for $\Phi_{ff}(\omega)$, the Fourier transform of the autocorrelation function, in terms of $F(\omega)$, the Fourier transform of the signal $f$.

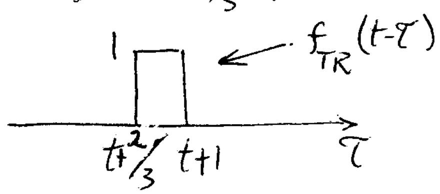
Note that the integral defining $\phi_{ff}(t)$ is a convolution, but not of $f$ and $f$. Let $f_{TR}(t-\tau) = f(\tau - t)$. Then clearly $\phi_{ff}(t) = (f * f_{TR})(t)$. Notice, however, that if $f_{TR}(t-\tau) = f(\tau - t)$, $\forall t$ or $\forall \tau$, then it must be that $f_{TR}(t) = f(-t)$, $\forall t$. That is, $f_{TR}$ denotes a time reversal of $f$. (hence our wise choice of the nomenclature $f_{TR}$ ☺). The function $f$ is real-valued $\Longrightarrow F^*(\omega) = F(-\omega)$. We also know $\mathcal{F}(f_{TR}(t)) = F(-\omega)$, by the time-reversal property of the CTFT $\Longrightarrow \Phi_{ff}(\omega) = F(\omega)F^*(\omega) = |F(\omega)|^2$

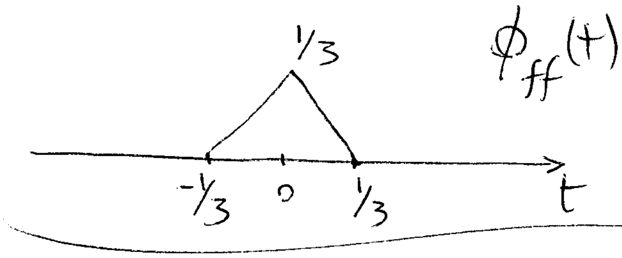
(b) Determine the autocorrelation function of the following signal $f$ characterized by:

$$f(t) = \begin{cases} 1 & \text{if } 2/3 \le t \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

$\phi_{ff}(t) =$  $*$ 







$t+1 < \frac{2}{3}$ $(t < -\frac{1}{3})$ $\longrightarrow$ no overlap between $f(\tau)$ and $f_{TR}(t-\tau)$

$1 < t + \frac{2}{3}$ $(\frac{1}{3} < t)$ $\longrightarrow$ no overlap between $f(\tau)$ and $f_{TR}(t-\tau)$

$t = 0 \longrightarrow$ maximal overlap between $f(\tau)$ and $f_{TR}(t-\tau)$.

$\longrightarrow$ For $-\frac{1}{3} < t < 0 \rightarrow$ linear increase

$0 < t < \frac{1}{3} \rightarrow$ linear decrease

LAST Name Correlate FIRST Name Auto

Lab Time Anytime

| Problem | Points | Your Score |
|---|---|---|
| Name | 10 | 10 |
| 1 | 60 | 60 |
| 2 | 60 | 60 |
| 3 | 50 | 50 |
| 4 | 20 | 20 |
| **Total** | **200** | 200 |