EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 3
Department of Electrical Engineering and Computer Sciences 29 April 2008
UNIVERSITY OF CALIFORNIA BERKELEY

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Lab Time 24/7

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" × 11" sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 6.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the six numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT3.1 (35 Points) In this problem, you'll develop the Parseval-Plancherel-Rayleigh Identity for discrete-time signals. Consider the discrete-time signals $x, y : \mathbb{R} \to \mathbb{C}$, which have corresponding Fourier transforms (DTFT) $X, Y : \mathbb{R} \to \mathbb{C}$. Let

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x(n)\, y^*(n) \qquad \text{and} \qquad \langle X, Y\rangle = \int_{\langle 2\pi\rangle} X(\omega)\, Y^*(\omega)\, d\omega\,.$$

(a) Prove the identity

$$\langle x, y\rangle = \frac{1}{2\pi}\langle X, Y\rangle\,, \quad \text{i.e.,} \quad \sum_{n=-\infty}^{\infty} x(n)\, y^*(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\, Y^*(\omega)\, d\omega\,.$$

$$\frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)Y^*(\omega)\,d\omega = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\left[\sum_{n=-\infty}^{\infty} y(n)e^{-i\omega n}\right]^* d\omega = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\sum_{n=-\infty}^{\infty} y^*(n)e^{i\omega n}\,d\omega$$

We now shamelessly interchange the infinite sum and the integral:

$$\frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)Y^*(\omega)\,d\omega = \sum_{n=-\infty}^{\infty} y^*(n)\left[\frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)e^{i\omega n}\,d\omega\right] = \sum_{n=-\infty}^{\infty} y^*(n)\,x(n)$$

(b) Use the result of part (a) to prove the identity $\displaystyle\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(\omega)|^2\, d\omega$.

Let $y(n) = x(n)$ in part (a)

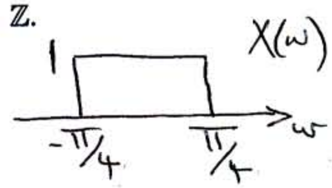
$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \sum_{n=-\infty}^{\infty} x(n)x^*(n) = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Equal, according to part (a)

$$\frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)Y^*(\omega)\,d\omega = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)X^*(\omega)\,d\omega = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(\omega)|^2\,d\omega$$

(c) Determine the energy $\mathcal{E}_x \triangleq \displaystyle\sum_{n=-\infty}^{\infty} x^2(n)$ of the real-valued discrete-time signal x, where $x(n) = \dfrac{\sin(\pi n/4)}{\pi n}$, $\forall n \in \mathbb{Z}$.

$$x(n) = \frac{\sin\left(\frac{\pi n}{4}\right)}{\pi n} \xleftrightarrow{\mathcal{F}}$$



$$\Rightarrow |X(\omega)|^2 = \begin{cases} 1 & |\omega| < \frac{\pi}{4} \\ 0 & e/w \end{cases} \;=\; X(\omega)$$

$$\mathcal{E}_x = \sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} |X(\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} d\omega = \frac{1}{2\pi}\left(\frac{\pi}{2}\right) = \frac{1}{4}$$

because $x(n) \in \mathbb{R}$, $\forall n$

part (b)

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$$\Rightarrow \boxed{\mathcal{E}_x = \frac{1}{4}}$$

MT3.2 (50 Points) We disclose bits of information about a discrete-time LTI filter H. Your task is to synthesize the hints and determine the filter completely. You receive partial credit for each hint that you synthesize correctly, but you must be clear about how you interpret and process each hint. Use the space immediately following each hint to show us your thinking process.

(I) The filter's frequency response H is real-valued; that is, $H(\omega) \in \mathbb{R}, \forall \omega \in \mathbb{R}$.

$H(\omega) \in \mathbb{R} \implies h(n) = h^*(-n)$ Impulse response is conjugate symmetric.

(II) The filter's impulse response h is real-valued; that is, $h(n) \in \mathbb{R}, \forall n \in \mathbb{Z}$.

$$\left.\begin{array}{l} h(n) \in \mathbb{R} \\ \text{Also, from (I): } h(n) = h^*(-n) \end{array}\right\} \implies h(n) = h(-n), \quad \forall n$$

Impulse response is real-valued and even.

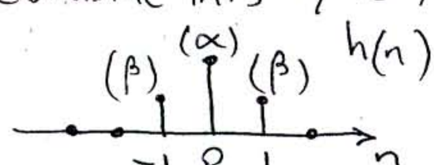
(III) A discrete-time LTI filter G has impulse response g and frequency response G. The filter G is causal and its frequency response is described by

$$G(\omega) = H(\omega)e^{-i\omega}, \quad \forall \omega \in \mathbb{R}.$$

$$\left.\begin{array}{l} G(\omega) = H(\omega)e^{-i\omega} \implies g(n) = h(n-1) \\ G \text{ is causal} \implies g(n) = 0 \quad n < 0 \end{array}\right\} \implies h(n-1) = 0 \quad n < 0$$

$\implies h(n) = 0 \quad n \leqslant -2$. Combine this w/ (II) $\implies$

$h(n) = 0, \quad n \geqslant 2 \implies$



The impulse response can have at most three non-zero values. By symmetry $h(1) = h(-1)$. The unknown values α, β are real.

(IV) The response of the filter H to the input signal x characterized by

$$x(n) = 1 + \cos\left(\frac{\pi}{4}n\right) + \cos(\pi n), \forall n \in \mathbb{Z}$$

is

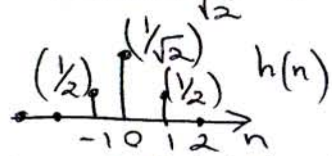
$$y(n) = \left(\frac{1}{\sqrt{2}} + 1\right) + \sqrt{2}\cos\left(\frac{\pi}{4}n\right) + \left(\frac{1}{\sqrt{2}} - 1\right)\cos(\pi n), \forall n \in \mathbb{Z}.$$

From this hint, we learn that $H(0) = \frac{1}{\sqrt{2}} + 1$ and $H(\pi) = \frac{1}{\sqrt{2}} - 1$ (b/c $\cos(\pi n) = (-1)^n$, which comes out as $\left(\frac{1}{\sqrt{2}} - 1\right)(-1)^n$). But $H(0) = \sum_{n=-\infty}^{\infty} h(n) = \alpha + 2\beta = \frac{1}{\sqrt{2}} + 1$

$H(\pi) = \sum_{n=-\infty}^{\infty} (-1)^n h(n) = \alpha - 2\beta = \frac{1}{\sqrt{2}} - 1$ $\Rightarrow \begin{cases} \alpha = \frac{1}{\sqrt{2}} \\ \beta = \frac{1}{2} \end{cases}$

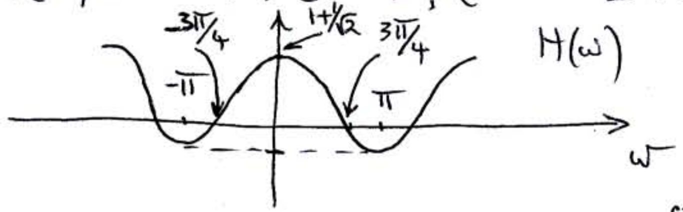
(a) Determine, and provide a well-labeled plot of, the impulse response h.

Solving the system of equations $\begin{cases} \alpha + 2\beta = \frac{1}{\sqrt{2}} + 1 \\ \alpha - 2\beta = \frac{1}{\sqrt{2}} - 1 \end{cases}$, we obtain the values $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \frac{1}{2}$



(b) Determine a simple expression for, and provide a well-labeled plot of, the frequency response $H(\omega)$, $\forall |\omega| \leq \pi$.

$H(\omega) = \alpha + \beta e^{i\omega} + \beta e^{-i\omega} = \alpha + 2\beta\cos\omega \Rightarrow \boxed{H(\omega) = \frac{1}{\sqrt{2}} + \cos\omega}$



(c) Determine a simple numeric value for $\int_{-\pi}^{+\pi} |H(\omega)|^2 \, d\omega$.

$$\int_{-\pi}^{\pi} |H(\omega)|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |h(n)|^2 = 2\pi\left(\alpha^2 + 2\beta^2\right) = 2\pi\left(\frac{1}{2} + 2\left(\frac{1}{4}\right)\right)$$

$$= 2\pi$$

By Parseval's Identity

$$\boxed{\int_{-\pi}^{\pi} |H(\omega)|^2 d\omega = 2\pi}$$

MT3.3 (20 Points) The *autocorrelation function* of a finite-energy discrete-time signal is defined as

$$\phi_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k)\,x(k+n)\,.$$

(a) Show that ϕ_{xx} is expressible as the convolution $\phi_{xx}(n) = (x * x_{\mathrm{TR}})(n)$, where x_{TR} is the time-reversed counterpart of x, i.e., $x_{\mathrm{TR}}(n) = x(-n)$, $\forall n \in \mathbb{Z}$.

$$(x * x_{TR})(n) = \sum_{m=-\infty}^{\infty} x(m)\, x_{TR}(n-m) = \sum_{m=-\infty}^{\infty} x(m)\; x(m-n)$$

Let $k = m-n \Rightarrow m = k+n$

$$x_{TR}(n) = x(-n) \Rightarrow x_{TR}(n-m) = x(-(n-m)) = x(m-n)$$

$$(x * x_{TR})(n) = \sum_{m=-\infty}^{\infty} x(k+n)\, x(k) = \phi_{xx}(n)$$

$$\phi_{xx}(n) = \sum_{k=-\infty}^{\infty} x(k)\, x(k+n) = (x * x_{TR})(n)$$

(b) Suppose $x(n) = a^n u(n)$, $\forall n \in \mathbb{Z}$ and $-1 < a < 1$. Determine a reasonably simple expression for $\Phi_{xx}(\omega)$, the DTFT of the autocorrelation function $\phi_{xx}(n)$.

$$x(n) = a^n u(n) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(\omega) = \frac{1}{1-ae^{-i\omega}}$$

$$\Phi_{xx}(\omega) = \mathcal{F}\{\phi_{xx}(n)\}$$

$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \qquad x_{TR}(n) = x(-n) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} x(-n)e^{-i\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{i\omega m} = X(-\omega)$$

$$\text{since } x(n) \in \mathbb{R} = \sum_{m=-\infty}^{\infty} x^*(m)e^{i\omega m} = X^*(\omega)$$

$$\phi_{xx}(n) = (x * x_{TR})(n) \Longleftrightarrow \Phi_{xx}(\omega) = X(\omega)\, X^*(\omega) = |X(\omega)|^2$$

$$X^*(\omega) = \frac{1}{1-ae^{i\omega}} \Longrightarrow \Phi_{xx}(\omega) = \frac{1}{1-ae^{-i\omega}} \cdot \frac{1}{1-ae^{i\omega}} = \frac{1}{1-ae^{i\omega}-ae^{-i\omega}+a^2}$$

$$\Longrightarrow \Phi_{xx}(\omega) = \frac{1}{(1+a^2) - 2a\cos\omega}$$

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Problem	Points	Your Score
Name	10	10
1	35	35
2	50	50
3	20	20
Total	**115**	115