CS174 Fall, 2002 Midterm 1

Except for Problem 8 you are only required to write down the answers. If you wish to show your work, do so in the space provided. If you need more space, use the back of the page and indicate on the front that you are doing so.

1. (9 points) A coin with the probability of heads p is tossed n times. What is the expected value of the number of heads minus the number of tails?

Answer:

2. (9 points) In a permutation = ((1), (2), ...,(n)), index i a called a *cumulative maximum* if (i) = max((1), (2), ..., (i)). What is the expected number of cumulative maxima in a random permutation of {1, 2, ..., n)?

Answer:

3. (9 points) Consider two urns. The first contains two white and seven black balls, and the second contains five white and six black balls. A person chooses an urn at random (each urn being equally likely) and pulls out a ball. What is the probability that the ball was drawn from the first urn, given that the ball is white?

Answer:

4. (9 points) What is the probability that a random permutation of ten elements has exactly 3 cycles: one of length 5, one of length 3, and one of the length 2? Hint: count the permutations with the specified property, then divide by 10!.

Answer:

5. (9 points) $4n$ balls are thrown into n bins. What is the probability that a collision occurs in bin 1? What is the limiting value of this probability as n tends to infinity?

Answer:

6. (9 points) Chebyshev's Inequality states that, if random variable X has mean μ and variance ^{2}, then P(|X - μ| ≥ k) ≤ $^2/k^2$. Show that this inequality is tight in the case μ = 10, ^{2}= 16, k = 8 by exhibiting a random variable X with mean 10, variance 16 and a specific distribution such that P(|X-10| ≥ 8) = $16/8^2$.

Answer:

7. (9 points) Let the random variable X_n be the number of heads in n tosses of a coin with probability of heads p. Let E_n be the probability that X_n is even. Give a formula expressing E_n in terms of E_{n-1}.

Answer:

8. (17 points) Call a set *odd* if it has an odd number of elements, and *even* otherwise. It is a fact that a set with n elements, $n \geq 1$, has exactly 2^{n-1} odd subsets and exactly 2^{n-1} even subsets. Using the probabilistic method, you are to prove that, given subsets $S_1, S_2, ... S_t$ of a finite set U, there exists a set S is a subset of U such that at least $t/2$ of the sets S_1 S, S_2 S, ..., S_t S are odd.

Describe each of the following steps in the proof:

(a) (4 points) Definition of the sample space.

(b) (4 points) Definition of the random variable X whose expectation is to be computed.

(c) (4 points) Representation of X as a sum of indicator random variables.

(d) (5 points) Computation of the expectation of each indicator random variable.

9. (20 points) A certain company has selected n candidates to interview for a vacant position. According to the local law, the candidates have to be interviewed in sequence, and each one has to be either offered the job or rejected immediately after the interview, before the next candidate is interviewed.

The company has decided to adopt the following strategy. Schedule the n candidates in a random order. For some value of k (to be determined), interview and reject the first k candidates. After that, offer the position to the first candidate who is better than all of the first k. (The position remains open if no such candidate exists). The company assumes that no two candidates are equally qualified, so there are no ties.

(a) (5 points) Suppose first that $k = n - 1$; i.e., all candidate but one are interviewed and rejected, and then the last one is hired if he/she is the best; otherwise the position remain open. Give the probability that the best candidate is hired in this case.

Answer:

(b) (5 points) For given k, and n, what is the probability that the best candidate is hired, given that the best candidate is in position m of the ordering?

Answer:

(c) (5 points) For given k and n, what is the probability that the best candidate is hired? Express your result in terms of harmonic numbers.

Answer:

(d) (5 points) Approximating H_t by $\ln t$ for any t, give the approximate value of the k which maximizes the chance of choosing the best candidate.

Answer: