CS 70 Discrete Mathematics and Probability Theory

Summer 2014 James Cook — Midterm 2

Thursday July 31, 2014, 12:40pm-2:00pm.

Instructions:

- Do not turn over this page until the proctor tells you to.
- Don't write any answers on the backs of pages (we won't be scanning those). There is an extra page at the end in case you run out of space.
- The exam has 10 pages (the last two are mostly blank).

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ______________, ______________ ________________
(last) (first) (signature)

PRINT your discussion section and GSI (the one you attend): ________________________________

Name of the person to your left: ________________________________

Name of the person to your right: ________________________________

Name of someone in front of you: ________________________________

Name of someone behind you: ________________________________

PRINT your name and student ID: ________________________________

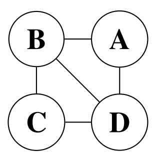
Short Answer

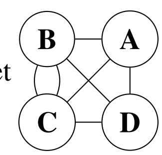
1. (4 pts.) You flip a fair coin three times independently. Conditioned on the event that you get at least one heads (H), what's the probability that it came up heads all three times (HHH)?

Solution: Let E_1 be the event that the coin comes up heads at least once, and E_3 the event that it comes up heads all three times. Our sample space is the set of all sequences of heads and tails, so it has $2^3 = 8$ elements. By simple counting, we see $\Pr[E_1] = \Pr[E_1 \cap E_3] = \frac{7}{8}$ and $\Pr[E_3] = \frac{1}{8}$, so $\Pr[E_3|E_1] = \frac{1/8}{7/8} = \frac{1}{7}$.

2. (4 pts.) Suppose G is a connected undirected graph with n nodes. Two of the vertices have odd degree, and the rest have even degree. (So G does not have an Eulerian tour.)

Now, we add two uniformly random edges e_1 and e_2 to G. (We don't allow self-loops but we allow multiple edges, so there are $\binom{n}{2}^2$ equally likely ways to choose e_1 and e_2.)

Example: if $n = 4$ and G is 

and our random edges are (A,C) and (B,C), we get 

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What is the probablility that the new graph has an Eulerian tour? Your answer should be a function of n.

Solution: The new graph can only have an Eulerian tour if all its vertices have even degree.

Let u and v be the two odd-degree vertices of G. Then u must be the endpoint of one of the two edges (but not both), and the same for v. So e_1 has the form (u,a) and e_2 has the form (v,b), or vice-versa. However, we cannot allow the nodes a and b to change from even to odd degree. So we must have $a = b$.

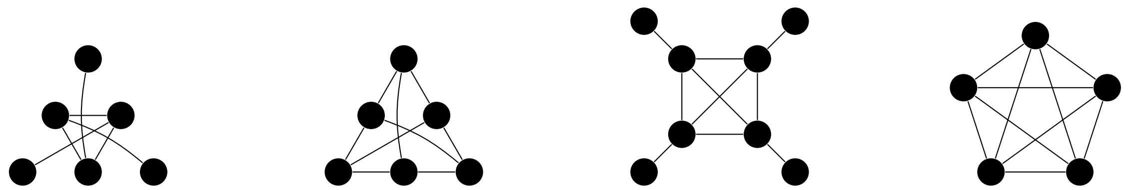
So e_1 and e_2 must either be $((u,a),(v,a))$ or $((v,a),(u,a))$ for some node a. There are $n-2$ choices for a, so this in $2(n-2)$ choices overall.

So the probability is $\dfrac{2(n-2)}{\binom{n}{2}^2}$.

Common mistake: Many students wrote $(n-2)/\binom{n}{2}^2$ instead of $2(n-2)/\binom{n}{2}^2$. Notice that our sample space considers $((u,a),(v,a))$ to be a different outcome from $((v,a),(u,a))$. (If we didn't distinguish these, there would be $\binom{\binom{n}{2}+1}{2}$ ways to choose e_1 and e_2 instead of $\binom{n}{2}^2$.)

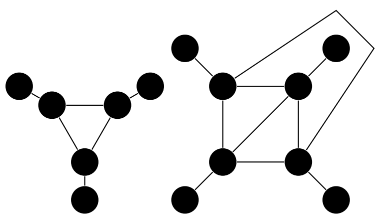
Short Answer (continued)

3. (4 pts.) Circle "Planar" below each planar graph and "Non-Planar" below each non-planar graph.



Planar / Non-Planar **Planar / Non-Planar** **Planar / Non-Planar** **Planar / Non-Planar**

Solution: The second and fourth graphs are $K_{3,3}$ and K_5, respectively, so are non-planar. The other two are planar:



4. (6 pts.) Prove the following statement using a combinatorial proof.

$$\binom{2n}{3} = \binom{n}{3} + \binom{n}{3} + n\binom{n}{2} + \binom{n}{2}n$$

Solution:

Both sides count the number of ways to choose 3 people from a group of n children and n adults. The left-hand side counts this directly: there are $2n$ people in all, and we're choosing a group of three. The right-hand side divides it into four cases:

- choose 3 of the n children (there are $\binom{n}{3}$ ways),
- choose 3 of the n adults (there are $\binom{n}{3}$ ways),
- choose 1 child and 2 adults ($n\binom{n}{2}$ ways), or
- choose 2 children and 1 adult ($\binom{n}{2}n$ ways).

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Counting

5. (12 pts.) Our new website, `cs70rules.com`, will require each student to choose a username satisfying the following two rules:

Rule 1. Usernames can only use upper-case letters A-Z. (So there are 26 possibilities for each character.)

Rule 2. Usernames must be six characters long.

For example, `STEVEN` is a valid username. `COOK85` and `PANDU` are not, because they violate Rules 1 and 2, respectively.

In the below questions, leave your answers as unevaluated expressions like "$\binom{20}{6} \times 7!$". You do not need to explain your answers.

(a) (2 pts.) How many possible usernames satisfy Rules 1 and 2?

(b) (2 pts.) Rule 3 is added: a username may not use a letter more than once. For example, `STEVEN` is no longer allowed, since it has two `E`s, but `JAMESC` is still allowed.

How many possible usernames are there that satisfy Rules 1-3?

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Counting (continued)

(c) (4 pts.) Rule 4 is added: the letters in a username must appear in alphabetical order. So `JAMESC` is not valid any more, but putting the letters in order, `ACEJMS` is valid. Similarly, `FEDCBA` is not valid but `ABCDEF` is valid.

How many possible usernames satisfy Rules 1-4?

(d) (4 pts.) After students start complaining, we remove Rules 3 and 4. We also relax Rule 1 so that a username can have up to one underscore (_).

For example, `AJAY_T` and `STEVEN` are allowed, but `I_AM_J` is not allowed, because it has more than one underscore.

How many possible usernames are there now?

Solution:

(a) 26^6. (26 choices for each letter.)

(b) $\frac{26!}{20!}$. (26 choices for the first letter, then 25 for the second to be different from the first, etc.)

(c) $\frac{26!/20!}{6!} = \binom{26}{6}$. (There is one username like this for every set of six different letters.)

(d) $26^6 + 6 \cdot 26^5$ (26^6 usernames without an underscore. With an underscore, there are 6 places to put the underscore, and then 26 choices for each of the five remaining letters).

Probability

6. (12 pts.) Pat has a deck with five cards, numbered [1] [2] [3] [4] [5]. She proposes the following game to Gary:

1. Gary takes two cards uniformly at random: call them A and B. Gary sees the number on A, but B is hidden so he can't see it.
2. Next, Pat takes a card C uniformly at random from the 3 that remain. (Gary doesn't see this either.)
3. Finally, Gary must choose whether to keep card A or B.

Whoever has the higher card wins.

Here's an example game. In Step 1, Gary takes $A = \boxed{3}$ and $B = \boxed{5}$. In Step 2, Pat takes $C = \boxed{4}$. Gary only knows that card A is $\boxed{3}$, and decides to keep A. Gary loses because $\boxed{4}$ is higher than $\boxed{3}$.

(a) (2 pts.) Gary models the game with a probability space. An outcome consists of the values of cards A, B and C. For example, one possible outcome is $(3,5,4)$, meaning Gary receives $A = \boxed{3}$ and $B = \boxed{5}$ and Pat recieves $C = \boxed{4}$.
If the probability distribution is uniform, then what is the probability of each outcome?

(b) (4 pts.) If Gary's strategy is to always keep card A, what's the probability that Gary wins?

Continued on the next page!

Probability (continued)

(c) (6 pts.) Gary tries a new strategy: if card A is $\boxed{5}$, then he keeps card A, and otherwise keeps card B. What is the probability that Gary wins with this new strategy?

Solution:

(a) There are $5 \times 4 \times 3 = 60$ outcomes, and we're working with the uniform distribution, so every outcome has probability $\frac{1}{60}$.

(b) Let W_1 be the event that Gary wins. This consists of all outcomes (A, B, C) where $A > C$. There is a one-to-one correspondance between outcomes in W_1 and outcomes in $\overline{W_1}$: switch the values of A and B. So W_1 and $\overline{W_1}$ have the same number of elements, so $\Pr[W_1] = \frac{1}{2}$.

(c) Let W_2 be the event that Gary wins with the new strategy.
Gary wins when card A is $\boxed{5}$, and also when card A is not $\boxed{5}$ but card B is higher than card C. There are 4×3 ways card A can be $\boxed{5}$, and among the $4 \times 4 \times 3$ ways card A can be not $\boxed{5}$, in half of them it is the case that card B is bigger than card C (by the same reasoning as in part (c)).
So $|W_2| = 4 \times 3 + \frac{1}{2} \times 4 \times 4 \times 3 = 36$, so Gary's probability of winning is $\frac{36}{60} = \frac{3}{5}$.

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Conditional Probability

7. (9 pts.) Suppose there are two events, A and B. You are given the following information:
$\Pr[A|B] = \frac{1}{2}$, $\Pr[B|A] = \frac{1}{3}$, $\Pr[A] = \frac{2}{3}$.

(a) (3 pts.) What is $\Pr[A \cap B]$?

(b) (4 pts.) What is $\Pr[B]$?

(c) (2 pts.) Are A and B independent events?

Solution:

(a) $\Pr[A \cap B] = \Pr[B|A]\Pr[A] = \frac{2}{9}$.

(b) Here's one way to solve this. Bayes' rule says

$$\Pr[A|B] = \frac{\Pr[B|A]\Pr[A]}{\Pr[B]}.$$

Solving for $\Pr[B]$,

$$\Pr[B] = \frac{\Pr[B|A]\Pr[A]}{\Pr[A|B]} = \frac{4}{9}.$$

(c) No. $\Pr[A \cap B] = \frac{2}{9}$, but $\Pr[A] \cdot \Pr[B] = \frac{2}{3} \cdot \frac{2}{9} = \frac{8}{27}$.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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[Doodle page! Draw us something if you want or give us suggestions or complaints.]