EE 121 Midterm 1 Solutions

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Kiran

1. (a) $\text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y = \mathbb{E}[X(X+Z)] - 0 = \mathbb{E}X^2 + \mathbb{E}[XZ] = \sigma_X^2$ as X and Z are uncorrelated, $\mathbb{E}[XZ] = 0$.

   (b)Yes. The covariance has the unit of power and hence depends on the unit of measure. The correlation coefficient $\rho_{XY} = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{\sigma_X}{\sqrt{\sigma_X^2\sigma_Z^2}}$ is independent of the unit of measure.

   (c) Following the derivation in (a), $\text{Cov}(X,Y) = \sigma_X^2 + \rho_{XZ}\sigma_X\sigma_Z$. If ρ_{XZ} is close to 1, then X and Z "add" up resulting in a higher correlation between X and Y. If ρ_{XZ} is small, then, X and Z are close to being independent and the correlation between X and Y is only through X. If ρ_{XZ} is close to -1, X and Z "null" each other and the correlation between X and Y decreases.

2. The estimation is an averaging function over elements whose lengths are proportional to the length of the data window. Hence, larger the data window, higher the complexity of estimation. However, as seen in the homework, the accuracy increases with increasing N, approaching the autocorrelation function as $N \to \infty$.

3. (a) (i) Uniquely decodable : Reconstruction from coded sequence is unambiguous. For all source symbols x_i, x_i' and $n, n' \in \mathbb{N}$, $\mathcal{C}(x_1)\mathcal{C}(x_2)\ldots\mathcal{C}(x_n) \neq \mathcal{C}(x_1')\mathcal{C}(x_2')\ldots\mathcal{C}(x_{n'}')$.

   (ii) Prefix free : No codeword is a prefix to another codeword.

   Prefix free $\Rightarrow$ uniquely decodable — the parsing is determined once the codeword ends. Uniquely decodable $\not\Rightarrow$ prefix free — a suffix free code is uniquely decodable.

   (b) (i) 001000001_1_100000. Therefore, the runs of zeros are of length, 2,5,0,0,5^+.

   (ii) From class, concatenation of prefix-free codes is also prefix free.

   (iii) This is a much more efficient scheme for small ϵ. It approaches $H(Y)$ faster and does not waste space for storing long codewords. Also, this has a lower complexity of encoding-decoding than the Huffman coding scheme.

   (iv) $P(Y_n = k) = P(\text{there are } k \text{ zeros followed by a 1}) = (1-\epsilon)^k\epsilon = p_k$, which is a geometric distribution. Clearly, Y_n's are identically distributed. Since they depend on non-overlapping intervals of X_i, they are independent, thus constituting an iid process.

   (v) The entropy of Y is $H(Y) = -\sum_{k=0}^{\infty} p_k \log p_k$ and $H(Y) \leq \bar{L} < H(Y) + 1$.

   (vi) Let $\mathcal{C}(Y_i)$ denote the encoded sequence of the run Y_i and $\mathcal{L}(Y_i)$ denote the length of run Y_i (inclusive of the 1's — a run of k zeros will count for a length of $k+1$). If we have M such runs, then, the compression ratio is given by,

$$\eta = \frac{\sum_{m=1}^{M} \mathcal{L}(\mathcal{C}(Y_i))}{\sum_{m=1}^{M} \mathcal{L}(Y_i)}$$

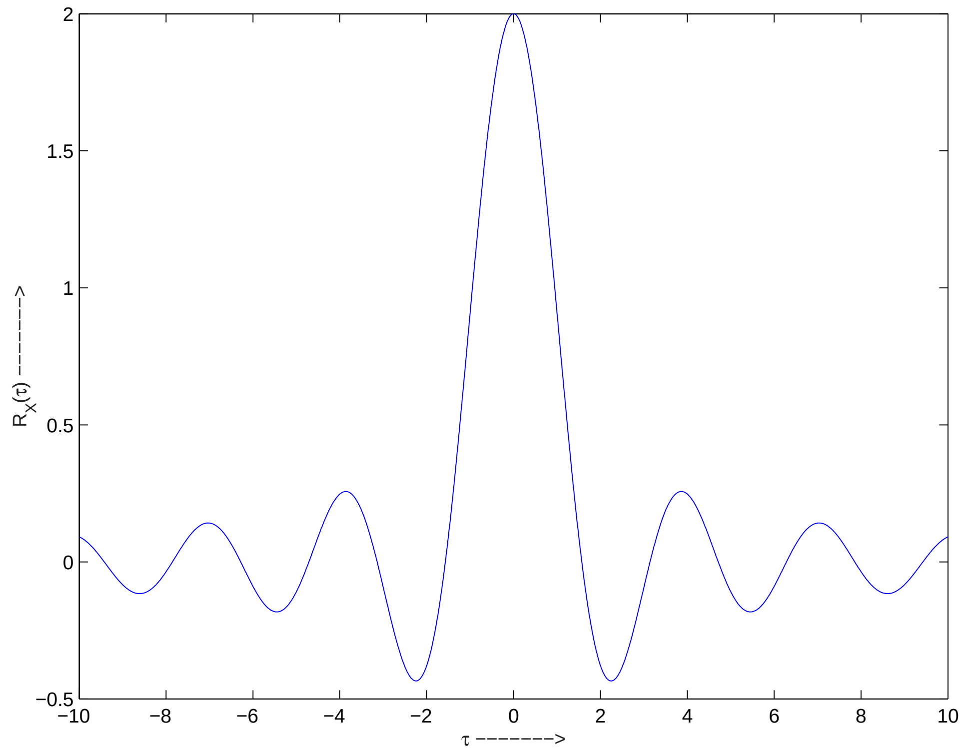

Figure 1: Covariance function for $X(t)$.

As $M \to \infty$, $\frac{1}{M}\sum_{m=1}^{M} \mathcal{L}(Y_i) \to \mathbb{E}Y+1$ (Law of Large Numbers), where, Y is a geometric $(1-\epsilon)$ random variable and from (c), $\frac{1}{M}\sum_{m=1}^{M} \mathcal{L}(\mathcal{C}(Y_i)) \to \bar{L}$, the average length of compressed data. Therefore, the compression ratio $\eta = \epsilon\bar{L}$.

4. (a) $R_X(\tau) = 2W\text{sinc}(2W\tau)$. This is plotted in Figure 1 for $W = 2$.

   (b) The process is sampled at Nyquist rate. $R_Y[k] = R_X(2WkT) = 2W\delta[k]$. Hence, $S_Y(\omega) = 2W$. Please verify this by drawing a picture in frequency domain also. The sampled process is a discrete white process.

   (c) A representative figure is shown in Figure 2. As $T \to \infty$, there are T portions each of height $\frac{2W}{T}$ overlapping, thus making the overall added portion approximately flat, of height $2W$. Therefore, the sampled process becomes white as $T \to \infty$.

   (d) The actual shape of $S_X(f)$ (assumed to be bounded) will affect the minor fluctuations within the flat region and how fast it becomes flat. It does not change the fact that the under-sampled process becomes white as $T \to \infty$.

5. I (a) This is shown in Figure 3 and the projection is $\left\langle \frac{\mathbf{u}}{\|\mathbf{u}\|}, \mathbf{v} \right\rangle \frac{\mathbf{u}}{\|\mathbf{u}\|}$.

   (b) Every vector in S can be written as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. Therefore, if we denote $a_1\mathbf{u}_1 + a_2\mathbf{u}_2$ as the projection of $\mathbf{v}$ onto S, the vector, $\mathbf{v} - (a_1\mathbf{u}_1 + a_2\mathbf{u}_2)$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. Solving, we have that the projection of $\mathbf{v}$ onto S is, $\frac{<\mathbf{v},\mathbf{u}_1>}{<\mathbf{u}_1,\mathbf{u}_1>}\mathbf{u}_1 + \frac{<\mathbf{v},\mathbf{u}_2>}{<\mathbf{u}_2,\mathbf{u}_2>}\mathbf{u}_2$.

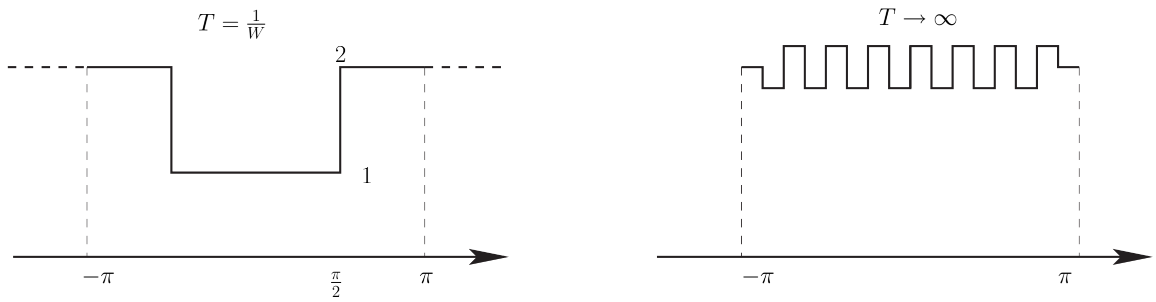

Figure 2: Effect of sampling below Nyquist rate

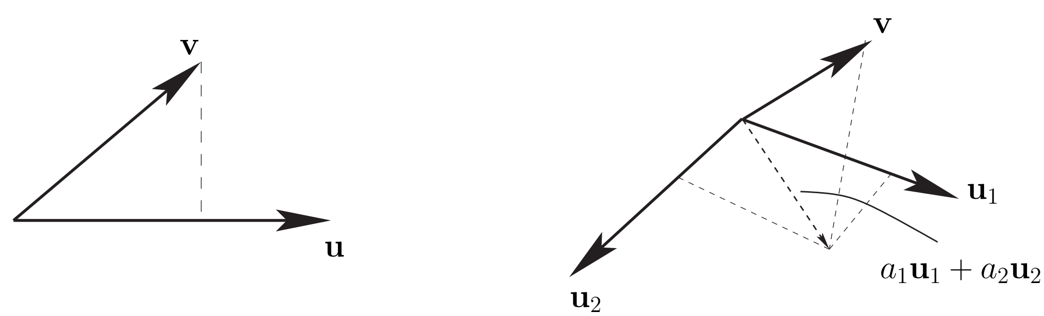

Figure 3: Projections and vector spaces

(c) define a new vector $\mathbf{u}_2' = \mathbf{u}_2 - \frac{<\mathbf{u}_1,\mathbf{u}_2>}{\mathbf{u}_1,\mathbf{u}_1>}\mathbf{u}_1$. Now, $\mathbf{u}_1$ and $\frac{\mathbf{u}_2'}{\|\mathbf{u}_2'\|}$ form an orthonormal basis for S and we can use results in (b).

(d) The projection on S depends only on the "angle" between $\mathbf{v}$ and S and hence independent of the basis chosen.

II (a) A signal $x(t)$ band-limited to $(-W, W]$ can be reconstructed from its samples at $t = nT$, iff $T \geq \frac{1}{2W}$. For $T = \frac{1}{2W}$, the reconstruction is $x(t) = \sum_{n=-\infty}^{\infty} x(nT)\text{sinc}(2W(t-nT))$.

(b) From (a), all band-limited functions can be written as linear combination of $\{\sqrt{2W}\text{sinc}(2W(t-nT))\}_{n=-\infty}^{\infty}$. Since these are orthonormal, they form an orthonormal basis. The dimension of this vector space is infinite.

(c) Here, $y(t)$ is not necessarily bandlimited. Let $y_b(t)$ denote $y(t)$ passed through a $(-W, W]$ low-pass filter. Then, the projection of $y(t)$ onto $\mathcal{V}$ is $y_b(t)$. To prove this, consider the projection of $y(t)$ onto one of the orthonormal basis,

$$\begin{aligned}
< y, \sqrt{2W}\text{sinc}(2W(t-nT)) > &= \sqrt{2W}\int y(t)\text{sinc}(2W(t-nT))\ dt \\
&= \sqrt{2W}\int Y(f)\text{Rect}(-W, W]e^{-j2\pi nfT}\ df \\
&= \sqrt{2W}\int Y_b(f)e^{-j2\pi nfT}\ df \\
&= \sqrt{2W}\int y_b(t)\delta(t-nT)\ dt = \sqrt{2W}y_b(nT)
\end{aligned}$$

Therefore, the projection of $y(t)$ onto $\mathcal{V}$ is $\sum y_b[nT]\text{sinc}(2W(t-nT))$ which is $y_b(t)$ from sampling theorem.

(d) The noise out of band does not affect the signal subspace. Since the noise is White Gaussian, the projection of $z(t)$ on a basis out of $(-W, W]$ is independent of the noise within $(-W, W]$ and hence provides no information about noise within $(-W, W]$ band — the out-of-band noise is on an orthogonal subspace. Therefore, the projection of noise on V is a sufficient statistic for detecting the transmitted message.

(e) In the first approach, we can use infinitely many matched filters, matched to each one of the orthonormal basis, as shown in Figure 4(a). However, each of matched filters are shifted versions of $\sqrt{2W}\text{sinc}(2Wt)$. Therefore, from the results in class, we can obtain the sufficient statistics by using this filter and sampling the output at $\{t = nT\}_{n=-\infty}^{n=\infty}$, shown in Figure 4(b). (can you point out which sample here corresponds to the output of the matched filters above?)

Alternatively, from part (d), we have that all the relevant information is present within the band of interest, $(-W, W]$. Hence, we can filter the received signal with a low-pass filter of bandwidth $(-W, W]$ and extract all the relevant signal and noise information. Since this is bandlimited, sampling the filtered waveform at $\{t = nT\}_{n=-\infty}^{\infty}$ provides all information for reconstruction of the original signal and therefore provides the sufficient statistics for detection. This is shown in Figure 4(c) (Can you argue/prove that figures (b) and (c) are the same?)

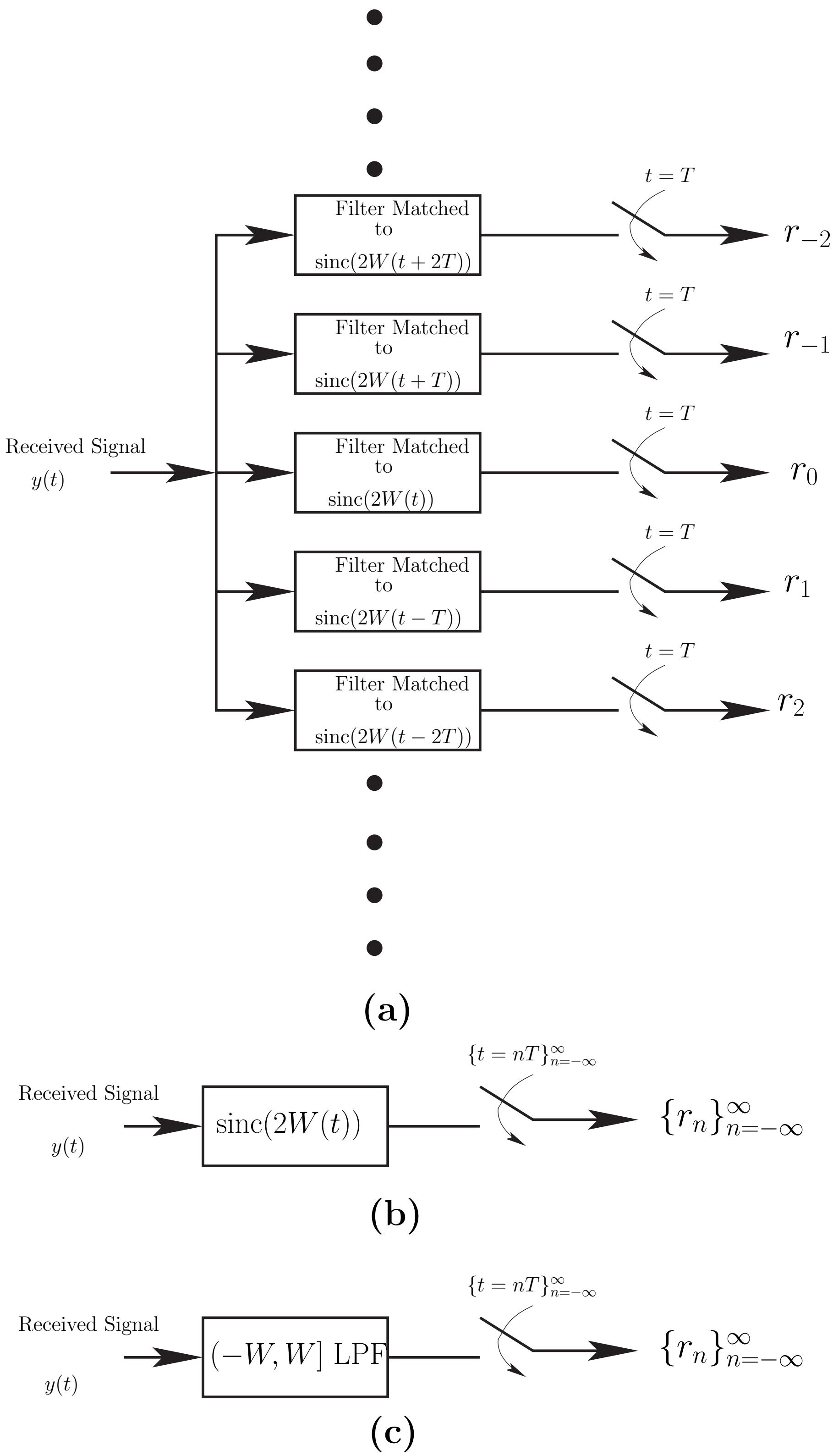

Figure 4: Problem 6